Valuing Contracts with Payoffs Based on Realized Volatility

Realized volatility (RV) forward contracts allow users to take a position whose payoff depends directly on the future volatility of an index. You can replicate the payoff of an RV contract by means of a portfolio of standard index options and futures. The fair value of the contract is the cost of implementing this replicating strategy. Because the strategy involves options, the fair forward realized volatility of the index turns out to be close to current implied volatilities near the money. For an example of an RV transaction on the S&P 500, see the section on ***Highlight Uses of Derivatives****.*

Introduction

Users of standard index options have exposure to both the underlying level of the index and its volatility. Sometimes, investors want straightforward exposure to the future volatility of the index itself. Realized volatility (RV) contracts allow users to take a position whose payoff depends directly and only on the future volatility of the index. In this note we focus on RV forward contracts.

An RV forward contract is defined to pay at expiration the difference in dollars between the actual return volatility realized by the index over the lifetime of the contract and some previously agreed upon "delivery" volatility[7]. This delivery volatility is analogous to the delivery price of a stock forward contract. The RV contract must also specify the precise method for calculating at expiration the realized volatility between contract inception and expiration; a natural choice is the standard estimate of daily volatility as the square root of the annualized variance of daily index returns, calculated from closing market levels.

To be specific, consider a one-year daily RV contract on the S&P 500 index with a delivery volatility of 14% and a notional amount of $100 per volatility point. Suppose that at the end of one year, the computed realized volatility turns out to be 16%. In this event, the holder of the long position in the RV forward contract will receive $200 from the holder of the short position. If instead, the computed realized volatility turns out to be 12%, the long holder will pay the short holder $200.

Like other derivatives, the fair value of an index RV forward contract is the cost of replicating its payoff using other instruments. We will show that you can replicate the payoff of an RV forward contract by means of a

7. Such a contract has a "notional amount" of $1, because it pays $1 per volatility point.

portfolio of standard index options and futures; the options component of the replicating portfolio is static, and need never be adjusted, while the futures portfolio must be continually rebalanced. It may seem surprising that a combination of options and futures, each of whose payoff and value is sensitive to the index level, can be combined to create a payoff that depends on volatility but is insensitive to the index level.

The value of an RV forward contract is the cost of implementing this replicating strategy. Just as the forward price of a stock is defined to be the delivery price of a stock forward contract that makes its value zero, the *forward realized volatility* be defined as the delivery volatility that makes the RV forward contract have zero value. This represents the future realized volatility we can lock-in today. Because the replicating strategy involves options, the fair forward realized volatility often turns out to be close to the current implied volatility of near-the-money options. Holders of realized volatility contracts are therefore long the spread between current implied volatility and future realized volatility. For this reason, such contracts may be a natural vehicle for investors who want to take a position on the spread between today's implied volatility and future realized volatility.

Valuing and Hedging Realized Volatility Forward Contracts

Consider an RV forward contract on an index (such as the S&P 500) with a delivery volatility of K between now and the contract expiration at time T later. Denote the current index level by S, the continuously compounded annual riskless rate by r, the dividend yield by δ, and the annualized realized volatility over the life of the contract observed at expiration by Σ_T. The payoff of the contract at expiration is

$$(\Sigma_T - K) \quad \text{dollars}$$

where we have assumed a notional value of one dollar per volatility point.

For the purpose of the contract terms, Σ_T is defined as

$$\Sigma_T = 100\sqrt{\frac{V}{T}} \qquad \text{EQ. 1}$$

where V is the variance of observed daily index returns over the lifetime T of the contract; dividing by T annualizes the volatility, and multiplying by 100 expresses it in percentage points.

What is the value today of a contract with this payoff? It is the present value of the expected future realized volatility less the present value of the delivery volatility. In the spirit of risk-neutral options pricing, it is the present value of the expected future payoff, that is

$$exp(-rT)\,[F_{\Sigma} - K] \tag{EQ. 2}$$

where $F_{\Sigma} = E[\Sigma_T]$ denotes the expected value of the future realized volatility to expiration. A reasonable way to estimate F_{Σ} is to take the average of the forward local volatilities of the index over a simulation of all future index paths, since these are the future volatilities that can be *locked in* using currently available options. For more information on extracting local volatilities from index options prices, see Derman, Kani and Zou[8].

While this approach suggests a price for the contract, it doesn't explain how to replicate it. The key to replicating the volatility contract is the observation that, for an index that evolves continuously, there is a strict relation between realized variance V and index level S_t at time t, namely[9]

$$\frac{V}{2} = \int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0} \tag{EQ. 3}$$

The first term in this equation represents the total return over the life of the contract obtained from a long position in $1/S_t$ shares of the index, continually rebalanced as S_t changes. The second term represents a short position in a *log contract*[10], a derivative security that pays off an amount related to the logarithm of the return on the index at expiration. According to Equation 3, double the value of a portfolio of these two positions is the replication value of the variance V. Each of these positions can be valued by standard options pricing techniques to give the present value

$$\langle V\rangle = 2e^{-rT}(r-\delta)T - 2P_{\log}$$

8. Derman E., I. Kani and J. Z. Zou. *The Local Volatility Surface*. Quantitative Strategies Research Notes, December 1995.
9. Strictly speaking, this relation is only true when the variance V is calculated from continuously observed returns, but daily returns are a good enough approximation.
10. See for example A. Neuberger, *The Log Contract*, The Journal of Portfolio Management, Vol 20, no 2 (1994).

where $\langle V \rangle$ is the present value (or discounted expected value) of variance V and $P_{\log}$ is today's market value of the log contract.

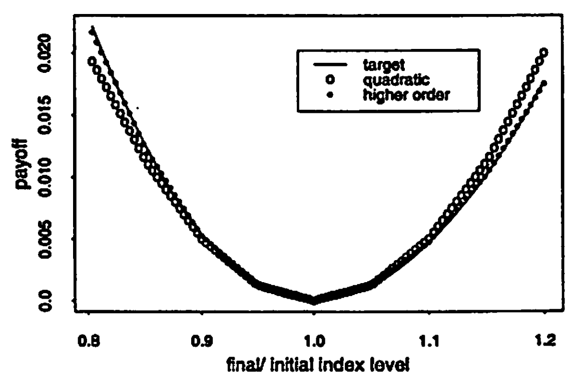
Since there is no traded log contract security, we cannot get $P_{\log}$ from the market. However, we can create this security synthetically, as follows.

Mathematically, a logarithmic payoff can be well approximated by the expression

$$\log\frac{S_T}{S_0} \approx \left(\frac{S_T - S_0}{S_0}\right) - \frac{1}{2}\left(\frac{S_T - S_0}{S}\right)^2 \qquad \text{EQ. 4}$$

when S_T is not too far from the initial value S_0. The first term in Equation 4 is proportional to the final index level S_T, and can be exactly replicated by a long position in $1/S_0$ forward contracts with index delivery price S_0, whose net current value is $e^{-\delta T} - e^{-rT}$. The second term's payoff increases quadratically with the deviation of the final index level from the initial, and can be replicated by a portfolio of standard calls and puts with the same expiration, and with strikes evenly spaced about S_0 [11]. Exhibit 21 shows that the quadratic and higher order pieces of the logarithmic payoff can be accurately replicated in this way.

EXHIBIT 21
Replication of the Quadratic and Higher Order Pieces of the Logarithmic Payoff using Options



11. We can fit the quadratic piece of the logarithmic payoff with an evenly spaced strip of out-of-the-money calls and puts with equal weights in addition to at-the-money calls and puts each with half the weight. To fit the higher order pieces of the logarithmic payoff we need to adjust the weights so that higher strike options will have somewhat smaller weights than lower strike options.

We can therefore think of the value of the log contract as approximately equal to the value a forward contract on the index itself, less the value of the portfolio P_{cp} of evenly spaced standard calls and puts, that is

$$P_{\log} = e^{-\delta T} - e^{-rT} - P_{cp} \qquad \text{EQ. 5}$$

We can combine Equations 4 and 5 to obtain the fair value of the variance V as

$$\langle V \rangle = 2e^{-rT}(r-\delta)T - 2(e^{-\delta T} - e^{-rT} - P_{cp}) \qquad \text{EQ. 6}$$

From Equation 1, we obtain the fair value of the realized volatility Σ_T as

$$\langle \Sigma_T \rangle = 100\left\{\frac{2}{T}[e^{-rT}(r-\delta)T + (P_{cp} + e^{-rT} - e^{-\delta T})]\right\}^{\frac{1}{2}} \qquad \text{EQ. 7}$$

In taking this final step, we have assumed that the square root of an expected value is the same as the expected value of the square root. This is not strictly correct, but it can be shown that the difference between the two is insignificant when the variance itself is relatively constant, which is the case in practice.

Equation 7 shows that the fair value of the realized future volatility is related to the value P_{cp} of series of strangles, which are the natural instruments for obtaining pure volatility exposure.

From Equation 2, the fair value of the RV forward contract is given by

$$\langle(\Sigma_T - K)\rangle = 100\left\{\frac{2}{T}[e^{-rT}((r-\delta)T + (P_{cp} + e^{-rT} - e^{-\delta T}))]\right\}^{\frac{1}{2}} - Ke^{-rT}$$

The choice of the options portfolio and its market price will in effect determine the price for the RV forward contract. More options will achieve better replication but may increase transaction costs.

An Example of a Fair Value Calculation

Assume the current index price S is 100, the continuously compounded annual interest rate r is 5% and the continuously compounded annual dividend yield δ is 2%. In addition, suppose that all options trade with implied volatilities equal to 15%. We want to calculate the forward 3-month daily realized volatility for this index. Since the implied volatility is flat, with no term or skew structure, we would intuitively expect that the fair value of realized volatility should also be equal to 15%.

Let us confirm this by replicating the quadratic piece of the logarithmic payoff in Equation 4 using a portfolio of out-of-the-money call and put options whose strikes range from 80% of the spot to 120% of the spot, in 5% intervals. We use equal weights for all options, except for the at the-money call and put options for which we use half that weight. Table 22 below shows the composition of this portfolio.

Table 22
A Portfolio of European Call and Put Options
Used for Calculating the Fair Value of Realized Volatility

Option Type	No. of Options $[10^{-4}]$	Strike	Security Value	Position Value $[10^{-4}]$
call	5	120	0.028	0.141
call	5	115	0.128	0.638
call	5	110	0.471	2.356
call	5	105	1.403	7.017
call	2.5	100	3.367	8.418
put	2.5	100	2.617	6.544
put	5	95	0.903	4.515
put	5	90	0.208	1.038
put	5	85	0.028	0.142
put	5	80	0.002	0.010
			Total Value: P_{cp}	30.820

From Equation 7 we find $<\Sigma_T> = 15.63\%$. We can improve the estimate of the realized volatility fair value by using an options portfolio with weights which also fit the higher order pieces of the logarithmic payoff. Table 23 shows the composition of this new portfolio.

Table 23
Improving the Fair Value Calculation by Adjusting the Option Weights

Option Type	No. of Options $[10^{-4}]$	Strike	Security Value	Position Value $[10^{-4}]$
call	3.0	120	0.028	0.084
call	3.5	115	0.128	0.447
call	4.0	110	0.471	1.885
call	4.5	105	1.403	6.315
call	2.5	100	3.367	8.418
put	2.5	100	2.617	6.544
put	5.5	95	0.903	4.967
put	6.0	90	0.208	1.246
put	6.5	85	0.028	0.185
put	7.0	80	0.002	0.014
			Total Value: P_{cp}	30.105

Using this table our improved estimate for the fair value of realized volatility is $<\Sigma_T> = 15.45\%$. To improve the fair value estimation even further we must use more strikes. Using strikes evenly spaced in 2% intervals and adjusted weights, the estimation of fair value improves to $<\Sigma_T> = 15.03\%$. In the limit of infinitely many strikes where the distance between the strikes approaches zero, the fair value of realized volatility will converge to 15%.

Effect of the Volatility Skew on Fair Value

Assume that instead of the constant volatility of 15% in our previous example the implied volatility changes by 2% for every 5% change in the strike level, with higher strikes having lower implied volatilities. The composition of the options portfolio remains the same as before. However, since the present value of the portfolio P_{cp} is affected by the skew our estimate for the fair value of realized volatility will be different. The results are shown in Table 24:

Table 24
Estimating the Realized Volatility Forward Value in the Presence of the Implied Volatility Skew

Option Type	Weight	Strike	Implied Volatility(%)	SecurityValue	Position Value
call	3.0	120	7.0	0.000	0.000
call	3.5	115	9.0	0.002	0.008
call	4.0	110	11.0	0.137	0.548
call	4.5	105	13.0	1.061	4.776
call	2.5	100	15.0	3.367	8.418
put	2.5	100	15.0	2.617	6.544
put	5.5	95	17.0	1.198	6.591
put	6.0	90	19.0	0.520	3.119
put	6.5	85	21.0	0.217	1.409
put	7.0	80	23.0	0.087	0.611
				Total Value: ($P_{cp} \times 10^4$)	32.024

The fair value calculated from this table is $<\Sigma_T> = 15.94\%$. If we use strikes in 2% intervals we find a better estimate of $<\Sigma_T> = 15.45\%$. This value is larger than the at-the-money implied volatility of 15%. The reason is that the value of the portfolio varies non-linearly with respect to the overall changes of implied volatilities of its component options.

NOTE: This piece was prepared by Emanuel Derman, Michael Kamal, Iraj Kani, and Joe Zou, Quantitative Strategies, Goldman, Sachs & Co.