Lecture 7: Local Volatility Continued

7.1 One More Remark About Static Hedging

We showed in Section 5.4.1 that, in a Black-Scholes world with zero drift, the fair value for a down-and-out call with strike K and barrier B is given by

$$C_{DO}(S, K) = C_{BS}(S, K) - \frac{S}{B} C_{BS}\left(\frac{B^2}{S}, K\right) \qquad \text{Eq.7.1}$$

You can write the payoff of the first term in Equation 7.1as

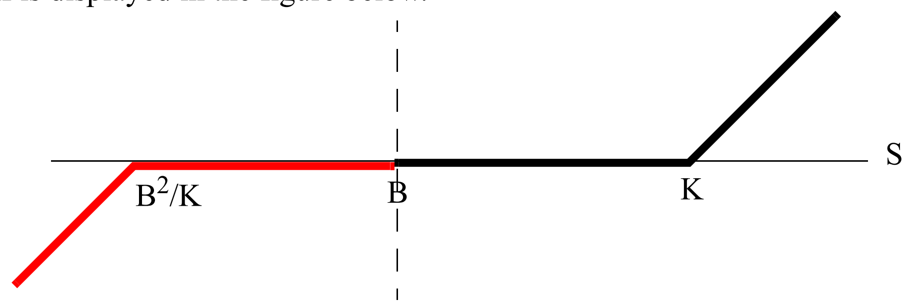
$$\theta(S-K)(S-K)$$

Similarly, you can write the payoff of $\frac{S}{B} C_{BS}\left(\frac{B^2}{S}, K\right)$ as

$$\frac{S}{B}\left(\frac{B^2}{S} - K\right)\theta\left(\frac{B^2}{S} - K\right) = \left(B - \frac{KS}{B}\right)\theta\left(\frac{B^2}{K} - S\right) = \frac{K}{B}\theta\left(\frac{B^2}{K} - S\right)\left(\frac{B^2}{K} - S\right)$$

This payoff represents that of *K/B* standard puts with strike B^2/K.

Thus, the payoff at expiration in Equation 7.1 is that of a long position in a call with strike *K* and a short position in *K/B* standard puts with strike B^2/K. The payoff is displayed in the figure below.

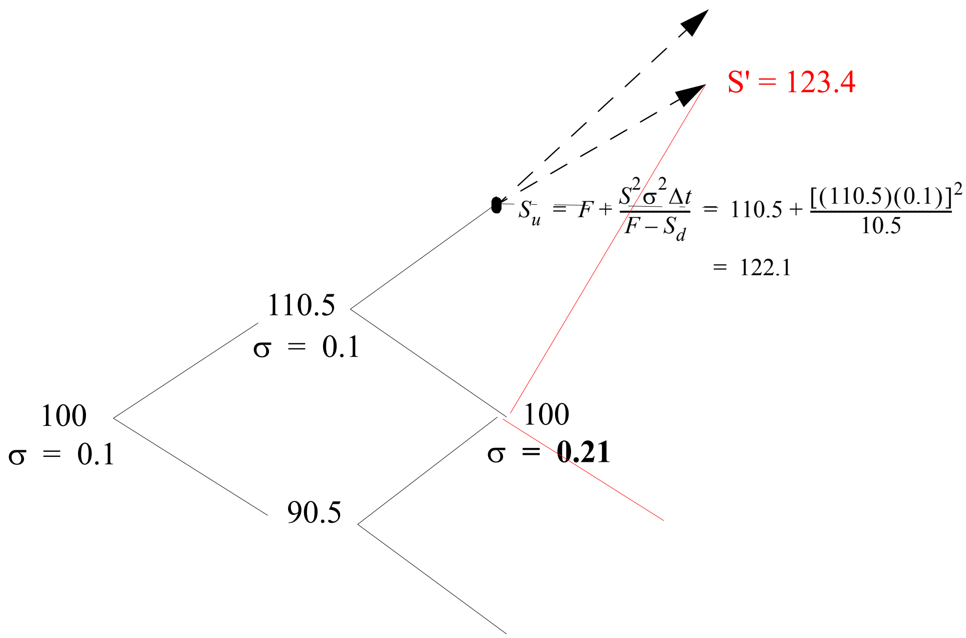


Roughly speaking the payoff of a down-and-out-call is that of an ordinary call and its price reflection (in log space) in the barrier. (In an arithmetic Brownian motion world, this would be accurate for price reflections rather than log price reflections.) You can see that the two payoff have positive and negative present discounted values that roughly cancel each other along the boundary B at all earlier times, and thus emulate the payoff of a down-and-out call with zero value on the barrier.This view is useful in thinking about the local volatilities that influence the option payoffs, which are sensitive to volatilities between B^2/K and *K*. Similar notions of payoff cancellation are useful in arriving at approximately static replicating portfolios for other barrier options

7.2 Difficulties with binomial trees

The positions of the nodes of the local volatility tree and the transition probabilities we discussed are uniquely determined by forward rates and the local volatility function we specify. But if the local volatility varies too rapidly with stock price or time, then, for finite Δt spacing between levels, you can obtain nodes with future stock prices that violate the no-arbitrage condition and result in binomial transition probabilities greater than 1 or less than zero.

Here is an example with $\Delta t = 1$ and $r = 0$.



We have chosen the local volatility on level 3 at stock price 100 to be 0.21 representing a very rapid rise from the value of 0.1 at the start of the tree. The S' node, the up node in level 4 relative to the central nodes, is technically the next down connected to S_u and should therefore lie below S_u, but in fact lies above it, and so violates the no-arbitrage condition: the up and down nodes emanating from S_u will both lie above the forward. These sorts of problems can be remedied by taking much smaller time steps Δt, but smaller time steps produce their own problems in calibrating a finely grained local volatility tree to a coarsely grained implied volatility surface, by trying to extract to extract more information that is available from implied volatilities. Therefore, it's sometimes easier to use trinomial trees, which we'll discuss later in this lecture. They provide greater flexibility in avoiding arbitrage situations.

7.3 Implied Trees and Calibration

We have shown how to build a local volatility binomial tree using $\sigma(S, t)$ and then determine the resulting surface of implied volatilities $\Sigma(S, t, K, T)$. In reality, however one is given discretely spaced implied volatilities $\Sigma(S, t, K_i, T_i)$ for discrete strikes K_i and expirations T_i, and one wants to calibrate a local volatility surface $\sigma(S, t)$ to these implied volatilities. This is the problem of building "implied trees" – an implied local volatility tree being a generalization of implied volatility. Implied volatility is a single variable defining a Black-Scholes tree; the implied tree is a representation of local volatility defining an evolution of a stock price consistent with options prices

Implied trees were first defined and developed in a binomial framework in the paper *The Volatility Smile and Its Implied Tree* which you can find on the course website, or published, as Derman, E. & I. Kani. "Riding on a Smile." RISK, 7(2) Feb.1994, pp. 139-145. Additional references are provided in that paper, and in *The Local Volatility Surface* (see my website too) and in James's book on Option Theory, to name just a few of many sources. The continuous-time derivation of the same results were derived by Bruno Dupire, *Pricing With A Smile*, RISK, January 1994, pages 18-20, and is also covered in the following sections of this lecture.

The key point of these papers is that there is a unique stochastic stock evolution process with a variable continuous volatility $\sigma(S, t)$ that can fit all options prices and their *continuous* implied volatilities $\Sigma(S, t, K, T)$. The tree representation of this stochastic process is called the *implied tree*.

The paper *The Volatility Smile and Its Implied Tree* shows that the local volatility at each node is determined numerically and non-parametrically from options prices. Finding local volatilities from implied volatilities is solving an "inverse problem" – going backwards from output to input — and is analogous to finding a potential in physics from viewing the way particles move under its influence. This is theoretically straightforward, it turns out, but is rather difficult in practice. It's an "ill-posed" problem. Beginning with sparse implied volatilities and interpolating them into a surface, and then naively seeking a fine-grained mesh of local volatilities leads to unstable solutions in which small changes in the interpolated input can cause dramatic changes in the output. In practice, it may be better to assume some parametric form for the local volatility function and then find the parameters that make the tree's option prices match as closely as possible the market's option prices.

In a later class we'll discuss some methods of calibrating local volatilities to implied volatilities.

7.4 Dupire's Equation for Local Volatility

The Dupire equation describes the relationship between implied and local volatility. Assuming continuous surfaces, you can derive the local volatility from the options prices (or their corresponding implied volatilities) in a simple way. Once you find the local volatilities from the implied volatilities surface, you can use them to build a binomial (or more general kind of) tree, and then use it to value options.

Recall the Breeden-Litzenberger formula that relates the price of a butterfly spread to the risk-neutral probability:

$$p(S, t, K, T) = e^{r(T-t)} \frac{\partial^2}{\partial K^2}(C(S, t, K, T)) \qquad \text{Eq.7.2}$$

where C is the market price of a call (and has nothing to do with the Black-Scholes model), and p is the density or risk-neutral probability function that tells you the no-arbitrage price $p(S, t, K, T)dK$ of earning \$1 if the future stock price at time T lies between K and $K + dK$.

The density function is determined by the second derivative of the market option price w.r.t strike K. One can do even better and find the local volatility $\sigma(S, t)$ itself from the market prices of options and their derivatives. For zero interest rates and dividends, the local volatility at the stock price K is given by:

$$\frac{\sigma^2(K, T)}{2} = \frac{\frac{\partial}{\partial T}C(S, t, K, T)}{K^2\frac{\partial^2 C}{\partial K^2}} \qquad \text{Eq.7.3}$$

This is Dupire's equation, the continuous version of the procedure we used to construct a local volatility binomial tree.

If interest rates are r rather than zero,

$$\frac{\sigma^2(K, T)}{2} = \frac{\frac{\partial}{\partial T}C(S, t, K, T) + rK\frac{\partial}{\partial K}C(S, t, K, T)}{K^2\frac{\partial^2 C}{\partial K^2}} \qquad \text{Eq.7.4}$$

Equation 7.3 and Equation 7.4 are the correct generalization of the notion of forward stock volatilities consistent with current implied volatilities.

7.4.1 Understanding the Equation

We can interpret Equation 7.3 in economic terms.

$$\frac{\partial C}{\partial T} = \frac{C(S, t, K, T + dT) - C(S, t, K, T)}{dT}$$

is proportional to an infinitesimal calendar spread for standard calls with strike K, i.e. long a call with expiration $T + dT$ and short a call with expiration T.

$$\frac{\partial^2 C}{\partial K^2} = \frac{C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K - dK, T)}{(dK)^2}$$

is, similarly, proportional to an infinitesimal butterfly spread for standard calls with strike K.

Therefore the local variance $\sigma^2(K, T)$ at stock price K and time T is proportional to the ratio of the price of a calendar spread to a butterfly spread. A calendar spread and a butterfly spread are combinations of tradeable options, and so the local volatility can be extracted from traded options prices (if they are available)!

We can understand this intuitively. A calendar spread

$$C(S, t, K, T + dT) - C(S, t, K, T)$$

has value proportional to the risk-neutral probability $p(S,t,K,T)$ of the stock moving from S at time t to K at time T, times the variance $\sigma^2(K, T)$ at K and T that is responsible for the adding option value as expiration increases from T to $T + dT$. In other words,

$$\text{calendar spread} \sim p(S, t, K, T)\sigma^2(K, T) .$$

But, according to the Breeden-Litzenberger formula in Equation 7.2, the probability

$$p(S, t, K, T) \sim \frac{\partial^2}{\partial K^2} C(S, t, K, T) \sim \text{butterfly spread}$$

So, roughly speaking, combing the two equations above, we have

$$\text{calendar spread} \approx \text{butterfly spread} \times \sigma^2(K, T)$$

or

$$\sigma^2(K, T) \approx \frac{\text{calendar spread}}{\text{butterfly spread}}$$

You can rewrite Equation 7.4 as

$$\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K} - \frac{\sigma^2}{2}K^2\frac{\partial^2 C}{\partial K^2} = 0 \qquad \text{Eq.7.5}$$

which looks much like the Black-Scholes equation with t replaced by T and S replaced by K. But, *very importantly*, note that whereas the Black-Scholes equation holds for any contingent claim on S, Equation is much more restrictive, and holds only for standard calls (or puts). The Black-Scholes equation relates the value of *any* option at S, t to the value of that option at $S + dS, T + dT$. Equation 7.5 in contrast, relates the value of a *standard* option with strike and expiration at K, T to the same option with strike and expiration $K + dK, T + dT$ when S, t is kept fixed.

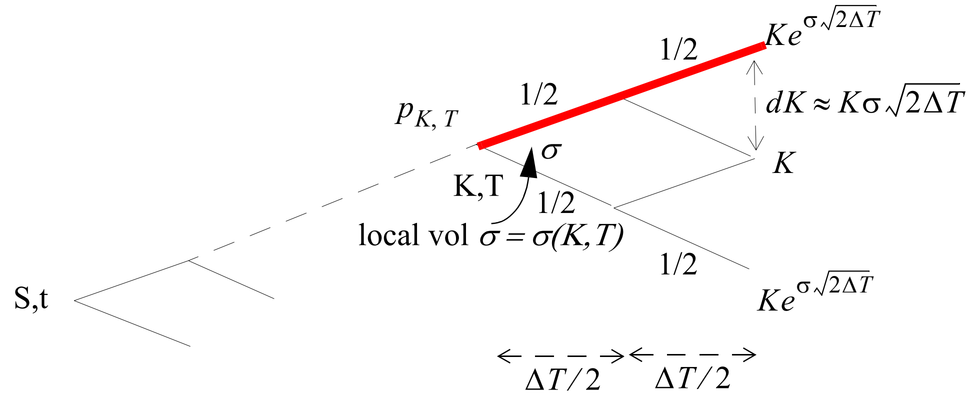
The value of this equation is that it tells you how to find $\sigma(K, T)$ and hence build an implied local volatility tree from options prices and their derivatives. You can then use that implied tree to value exotic options and to hedge standard options, knowing that you have one consistent model that values all standard options correctly rather than having to use several different inconsistent Black-Scholes models with different underlying volatilities.

This relation, or the implied tree in general, can also be useful for volatility arbitrage trading. You can calculate the future local volatilities implied by options prices and then see if they seem reasonable. If some of them look too low or too high in the future, you can think about buying or selling future butterfly and calendar spreads to lock in those future volatilities and make a bet on future volatility. In a later lecture we will discuss *gadgets*, the combinations of options that allow you lock bet on future volatility.

7.5 A Poor Man's Derivation of the Dupire Equation in a Binomial Framework.

Here is a poor man's derivation of the equation that provides intuition. A more rigorous derivation follows in the next section. You can find a more rigorous derivation that uses the Fokker-Planck or forward Kolmogorov equation in the appendix of the Derman-Kani paper on implied trees.

Let's use a Jarrow-Rudd tree that goes from (S,t) to (K,T) through two half-periods of time $\Delta T/2$,keeping interest rates zero for pedagogical simplicity.



The calendar spread obtains all its optionality from moving up the heavy red line in the last two periods:

$$C(S, t, K, T + dT) - C(S, t, K, T) \equiv \frac{\partial C}{\partial T}\Delta T = \overset{\text{probability}}{p_{K,T}\frac{1}{4}} \times \overset{\text{payoff}}{dK}$$

So, the value of the calendar spread per unit time is

$$\frac{\partial C}{\partial T} = \frac{1}{4}p_{K,T} \times \frac{dK}{dT} \qquad \text{Eq.7.6}$$

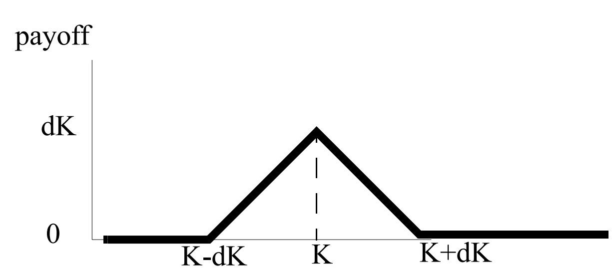
where $p_{K,T}$ is the risk-neutral probability of getting to the discrete binomial node (K, T), and $dK \approx \Delta K = K\sigma\sqrt{2\Delta T}$.

The butterfly spread

The risk-neutral probability $p_{K,T}$ of getting to the discrete binomial node (K, T) in terms of options payoffs is the value of a portfolio that pays $1 if the stock price is at node K, and zero for all other nodes at time T. The butterfly spread portfolio

$$C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K-dK, T)$$

has the payoff pattern



Therefore dividing this butterfly spread by dK produces a payoff which is \$1 at the node K and zero at adjacent nodes. Thus

$$p_{K,T} = \frac{C(S,t,K+dK,T) - 2C(S,t,K,T) + C(S,t,K-dK,T)}{dK} = \frac{\partial^2 C}{\partial K^2} dK$$

Now, from the Breeden-Litzenberger formula with zero interest rates

$$p(S,t,K,T) \equiv \frac{\partial^2 C}{\partial K^2}$$

so that

$$p_{K,T} = \frac{\partial^2 C}{\partial K^2} dK \qquad \text{Eq.7.7}$$

Combining the expression for $p_{K,T}$ in this equation with Equation 7.6 we obtain

$$\begin{aligned}\frac{\partial C}{\partial T} &= \frac{1}{4} p_{K,T} \frac{\Delta K}{\Delta T} = \frac{1}{4}\left(\frac{\partial^2 C}{\partial K^2}\right)\frac{(\Delta K)^2}{\Delta T} \\ &= \frac{1}{4}\left(\frac{\partial^2 C}{\partial K^2}\right)\frac{[K\sigma\sqrt{2\Delta T}]^2}{\Delta T} \\ &= \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}\end{aligned}$$

so that the local volatility $\sigma(K,T)$ is given by

$$\sigma^2(K,T) = \frac{\partial C}{\partial T} \Big/ \left(\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}\right)$$

You can regard this as a *definition* of the effective local volatility from options prices. We will show later how you can use this definition to construct *gadgets,* long positions in calendar spreads and short positions in butterfly spreads whose net cost is zero, that allow you to create forward contracts on local volatility.

7.6 A More Rigorous Proof of Dupire's Equation

This proof relies on stochastic calculus. First we write the stochastic PDE for the risk-neutral stock price:

$$\frac{dS_t}{S_t} = rdt + \sigma(S_t, t, .)dZ_t$$

where $\sigma(S_t, t, .)$ is the volatility, and may depend upon variables other than S_t.

Then the call value at time *t* is

$$C_t(K, T) = e^{-r(T-t)}E\{[S_T - K]_+\} \qquad \text{Eq.7.8}$$

where E denotes the q-measure risk-neutral expectation over S_T and all other stochastic variables.

We now examine the derivatives of the call value that enter the Dupire equation.

$$\frac{\partial C}{\partial K} = -e^{-r(T-t)}E\{\theta(S_T - K)\} \qquad \text{Eq.7.9}$$

$$\frac{\partial^2 C}{\partial K^2} = e^{-r(T-t)}E\{\delta(S_T - K)\} \qquad \text{Eq.7.10}$$

To find $\left.\frac{\partial C}{\partial T}\right|_K$ we need to take account of both the change in T and the corresponding change in S_T through Ito's Lemma. From Equation 7.8 we obtain

$$d_T C\big|_K = E\left\{\frac{\partial C}{\partial T}dT + \frac{\partial C}{\partial S_T}dS_T + \frac{1}{2}\frac{\partial^2 C}{\partial S_T^2}(dS_T)^2\right\}$$ expectation in risk-neutral measure over S_T

$$= E\left\{-rCdT + e^{-r\tau}\theta(S_T - K)dS_T + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(S_T, T, .)S_T^2 dT\right\}$$

$$= E\left\{-rCdT + e^{-r\tau}\theta(S_T - K)(rS_T dT) + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(S_T, T, .)K^2 dT\right\}$$

1

$$= E\left\{-re^{-r\tau}\theta(S_T - K)(S_T - K)dT + e^{-r\tau}\theta(S_T - K)rS_T dT + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(K, T, .)K^2 dT\right\}$$

$$= e^{-r\tau}E\left\{rK\theta(S_T - K)dT + \frac{1}{2}\delta(S_T - K)\sigma^2(K, T, .)K^2 dT\right\}$$

$$= -rK\frac{\partial C}{\partial K}dT + \frac{1}{2}E\{\sigma^2(K, T, .)\}\frac{\partial^2 C}{\partial K^2}K^2 dT$$

Then the change in the value of C when S_T and T change is given by

$$\left.\frac{\partial C}{\partial T}\right|_K = -rK\frac{\partial C}{\partial K} + \frac{1}{2}E\{\sigma^2(K, T, .)\}\frac{\partial^2 C}{\partial K^2}K^2$$

Rearranging terms, we obtain

$$E\{\sigma^2(K, T, .)\} = \frac{\left(\left.\frac{\partial C}{\partial T}\right|_K + rK\left.\frac{\partial C}{\partial K}\right|_T\right)}{\left.\frac{1}{2}\frac{\partial^2 C}{\partial K^2}\right|_T K^2} \qquad \text{Eq.7.11}$$

where we define the local volatility to be

$$\sigma^2(K, T) \equiv E\{\sigma^2(K, T, .)\} \qquad \text{Eq.7.12}$$

Thus, Equation 7.11, the Dupire equation.We have earlier shown that the denominator of Equation 7.11 is positive, and one can use dominance arguments to show that the numerator is too.

For another proof using the Fokker-Planck equation, see the Appendix of Derman, E. & I. Kani. “Riding on a Smile.” RISK, 7(2) Feb.1994, pp. 139-145.

7.7 An Exact Relationship Between Local and Implied Volatilities and Its Consequences

For zero interest rates and dividend yields, we derived

$$\sigma^2(K, T) = \frac{2\left.\frac{\partial C}{\partial T}\right|_K}{K^2\left.\frac{\partial^2 C}{\partial K^2}\right|_T}$$

If options prices are quoted in terms of their Black-Scholes implied volatilities Σ, then we can write

$$C(S, t, K, T) = C_{BS}(S, t, K, T, \Sigma(S, t, K, T))$$

where we continue assuming zero rates and dividends for simplicity.

By applying the chain rule for differentiation and the formulas for the Black-Scholes Greeks, one can show that

$$\sigma^2(K, T) = \frac{2\frac{\partial \Sigma}{\partial T} + \frac{\Sigma}{T-t}}{K^2\left(\frac{\partial^2 \Sigma}{\partial K^2} - d_1\sqrt{T-t}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{T-t}} + d_1\frac{\partial \Sigma}{\partial K}\right\}^2\right)} \qquad \text{Eq.7.13}$$

where $d_1 = \frac{\ln(S/K)}{\Sigma\sqrt{T-t}} + \frac{\Sigma\sqrt{T-t}}{2}$, and (don't forget!) that $\Sigma = \Sigma(S, t, K, T)$ is a function of *S, t, K, T*. When you take the chain rule to derive this you have to be very careful.

Equation 7.13 gives the local volatility directly in terms of the implied volatilities rather. All the Black-Scholes prices *C* in the numerator have cancelled those in the denominator. This formula is the generalization of the notion of forward volatilities in a no-skew world to local volatilities in a skewed world. We can now use this relation to justify some of our previous approximations in regarding local volatility as an average over implied volatilities.

We can now prove some of the previous relations we intuited between implied local volatility more precisely.

7.7.1 Implied variance is average of local variance if there no skew.

Let's look at Equation 7.13 when $\Sigma(S, t, K, T)$ is independent of strike K, with no skew at all. Then, writing $\tau = T - t$, we have

$$\frac{1}{2}\sigma^2(K, T) = \frac{\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma}{2\tau}}{K^2\frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{\tau}}\right\}^2} = \tau\Sigma\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma^2}{2}$$

We can rewrite this as

$$\sigma(\tau)^2 = \frac{\partial}{\partial\tau}(\Sigma^2\tau)$$

and so,

$$\tau\Sigma^2(\tau) = \int_0^{\tau}\sigma^2(u)du$$

the standard result that expresses the total variance as an average of forward variances.

7.7.2 Near the money, the slope of the skew w.r.t strike is 1/2 the slope of the local volatility w.r.t. spot.

In the complementary case $\Sigma = \Sigma(K)$ alone, independent of expiration, and $\frac{\partial\Sigma}{\partial\tau}$ is zero. Furthermore, let's assume a *weak linear* dependence of the skew on K, so that we keep only terms of order $\frac{\partial\Sigma}{\partial K}$, assuming that smaller terms of order $\left(\frac{\partial\Sigma}{\partial K}\right)^2$ or of order $\frac{\partial^2\Sigma}{\partial K^2}$ are negligible.

Then

$$\sigma^2(K,T) \approx \frac{\frac{\Sigma}{\tau}}{\frac{K^2}{\Sigma}\left(\left\{\frac{1}{K\sqrt{\tau}} + d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)} = \frac{\Sigma^2}{\left\{1 + d_1 K\sqrt{\tau}\frac{\partial\Sigma}{\partial K}\right\}^2}$$

and so

$$\sigma(K) = \frac{\Sigma(K)}{1 + d_1 K\sqrt{\tau}\frac{\partial\Sigma}{\partial K}} \qquad \text{Eq.7.14}$$

Close to at-the-money, $K = S + \Delta K$. Then

$$d_1 \approx \frac{\ln S/K}{\Sigma\sqrt{\tau}} \approx -\frac{\Delta K}{S(\Sigma\sqrt{\tau})} \approx -\frac{\Delta K}{K(\Sigma\sqrt{\tau})}$$

so that

$$\sigma(K) \approx \frac{\Sigma(K)}{1 - \frac{\Delta K}{\Sigma}\frac{\partial\Sigma}{\partial K}} \approx \Sigma(K)\left(1 + \frac{\Delta K}{\Sigma}\frac{\partial\Sigma}{\partial K}\right) \approx \Sigma(K) + (\Delta K)\frac{\partial\Sigma}{\partial K}$$

where ΔK is the distance from at the money. Therefore

$$\sigma(S+\Delta K) \approx \Sigma(S+\Delta K) + (\Delta K)\frac{\partial\Sigma}{\partial K}$$

and so, performing a Taylor expansion about S, we obtain

$$\frac{\partial}{\partial S}\sigma(S) = 2\left(\frac{\partial\Sigma}{\partial K}\right)\bigg|_{K=S} \qquad \text{Eq.7.15}$$

The local volatility $\sigma(S)$ grows twice as fast with stock price S as the implied volatility $\Sigma(K)$ grows with strike! Earlier in the course we derived this result more intuitively by regarding implied volatility as the average over local volatilities between spot and strike.

7.7.3 Implied volatility is an harmonic average over local volatility at short expirations.

One last thing about implied volatility as an average of local volatilities. Let's look at Equation 7.13 for very small times to expiration, and then derive a more careful averaging formula. (To fill in the details is a homework problem).

Here is the Equation 7.13 for zero rates and dividends:

$$\sigma^2(K,T) = \frac{2\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma}{\tau}}{K^2\left(\frac{\partial^2\Sigma}{\partial K^2} - d_1\sqrt{\tau}\left(\frac{\partial\Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{\tau}} + d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)}$$

Multiplying by τ in the numerator and denominator, we obtain

$$\sigma^2(K,T) = \frac{2\tau\frac{\partial\Sigma}{\partial\tau} + \Sigma}{K^2\left(\tau\frac{\partial^2}{\partial K^2}(\Sigma) - d_1\tau\sqrt{\tau}\left(\frac{\partial\Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left\{\frac{1}{K} + \sqrt{\tau}d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)}$$

In the limit as $\tau \to 0$, this becomes the o.d.e.

$$\sigma^2(K,T) = \frac{\Sigma}{K^2\left(\frac{1}{\Sigma}\left\{\frac{1}{K} + \sqrt{\tau}d_1\frac{d\Sigma}{dK}\right\}^2\right)} = \frac{\Sigma^2}{\left\{1 + \sqrt{\tau}Kd_1\frac{d\Sigma}{dK}\right\}^2}$$

Now $\sqrt{\tau}Kd_1 \to \frac{K\ln(S/K)}{\Sigma}$ as $\tau \to 0$, and we obtain

$$\sigma = \frac{\Sigma}{1 + \frac{K}{\Sigma}\frac{d\Sigma}{dK}\ln(S/K)}$$

Trasnforming from K into the new variable $x = \ln(S/K)$ we can rewrite this as the o.d.e.

$$\frac{\Sigma}{1 - \frac{x}{\Sigma}\frac{d\Sigma}{dx}} = \sigma \qquad \text{Eq.7.16}$$

In this lecture's homework you are asked to show that the solution is

$$\frac{1}{\Sigma(x)} = \frac{1}{x}\int_{o}^{x}\frac{1}{\sigma(y)}dy \qquad \text{Eq.7.17}$$

where $x = \ln(S/K)$. In other words, at very short times to expiration, the implied volatility is the harmonic mean of the local volatility as a function of $\ln S/K$ between spot and strike.

Equation 7.17 is intuitively reasonable, more sensible than an arithmetic mean. Suppose instead we used the formula $\Sigma(x) = \frac{1}{x}\int_{o}^{x}\sigma(y)dy$, an ordinary arithmetic mean. Also suppose that $\sigma(y)$ falls to zero above a certain level *S*, so that the stock price can never diffuse higher. Then the implied volatility of any option with a strike above that level should be zero. However, if the implied volatility were the arithmetic mean of the volatilities between the stock and the strike, its value would be non-zero, which is impossible if the stock can never reach the strike.

In contrast, for the harmonic mean in Equation 7.17, if $\sigma(y)$ becomes zero anywhere in the range between spot and strike, then the implied volatility for that strike, $\Sigma(x)$, becomes zero too, which is as to be expected.

There is an intuitive way to understand the presence of the harmonic mean.

$1/\sigma^2$ is the diffusion time through the medium for $\ln S$; the integral of this inverse variance to the strike level $\ln K$ is the total diffusion time. Equation 7.17 is roughly equivalent to saying that the total diffusion time is the sum of the local diffusion times. (Actually this applies to the square root of the diffusion times.) This is similar to the statement that, for a car travelling at a variable velocity over time, the total time for the trip is the sum of the local times; then, the average velocity is the total distance divided by the total time. The average velocity is *not* the average of the local velocities.[1]

1. This view has been taken by Avellaneda et al.

7.8 An Example Of Using The Dupire Equation To Calibrate the Local Volatilities

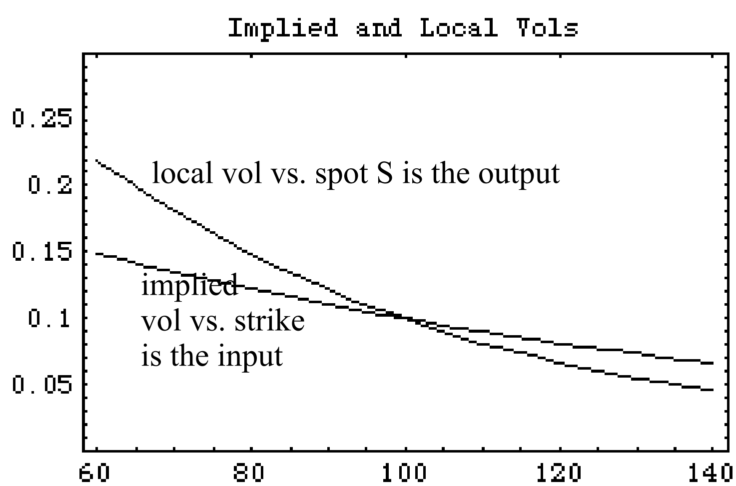
Although the graphs below look similar to some graphs in the previous lecture that compare implied and local volatilities, these are actually produced inversely. Here we are given a skew, and then we compute the local volatilities, i.e. we solve the inverse problem. In the previous lectures, we proceeded from local volatilities to implieds.

Let's assume an implied volatility skew of the form

$$\Sigma(K, T) = 0.1 \exp\left[-\left(\frac{K}{100} - 1\right)\right]$$

with no term structure of implied volatilities. At-the-money volatility is 10%, and then declines by approximately one volatility point for each 1 point rise in the strike.

I used Mathematica to evaluate the derivatives of the Black-Scholes options prices using this skew to calculate the local volatility in Equation 7.4. You can see in the figure below that the local volatility does increase roughly twice as fast with spot as implied volatility varies with strike.



Here is another more complex example where volatility varies with strike and expiration, increasing with expiration T but decreasing with strike K according to the formula

$$\Sigma(K, T) = (0.1 + 0.5T)\exp\left[-\left(\frac{K}{100} - 1\right)\right]$$

This smile has both term structure and skew. The graph below shows the local volatility computed in Mathematica from the Dupire equation at various spot levels one year in the future.

At one year in the future, the local volatility still has the characteristic variation with spot, but its value is higher. This illustrates our previous argument that if the term structure of implied volatilities is increasing, then local volatilities must grow about twice as fast with time as well as decrease twice as fast with spot

.