Lecture 6: Extending Black-Scholes; Local Volatility Models

Summary of the course so far:

- Black-Scholes is great but not perfect by any means.
- The smile violates it badly in all markets.
- The best approach is therefore to replicate - static if possible, else dynamic,
- Hedging errors and transactions costs mess things up. Which hedge to use? Implied volatility hedging leads to uncertain path-dependent total P&L; realized volatility hedging leads to a deterministic final P&L, but uncertain P&L along the way. Real life is more complex than either of these cases.
- You can strongly replicate any European payoff out of puts and calls, statically, independent of any valuation model. You can weakly replicate exotic options out of standard options, often only approximately. Weak replication needs a model that tells you the future smile.
- Some models for the smile: local volatility, stochastic volatility, jump diffusion.

This lecture:

- review of binomial models;
- the local volatility model.

6.1 The Binomial Model for Stock Evolution

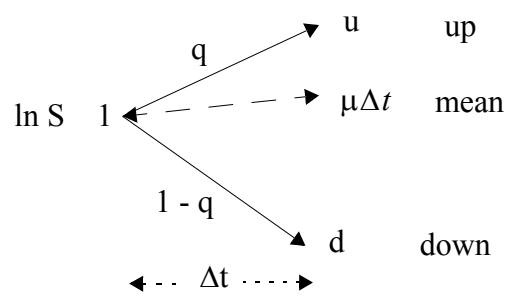
We intend to study ways of modifying the Black-Scholes model so as to accommodate the smile. It's easiest to begin in the binomial framework where intuition is clearer.

In the Black-Scholes framework, a stock with no dividend yield is assumed to evolve according to

$$d(\ln S) = \mu dt + \sigma dZ. \qquad \text{Eq.6.1}$$

The expected logarithmic return of the stock per unit time is μ; the expected return on the stock price, from Ito's lemma, is $\mu + \sigma^2/2$. The volatility of returns is σ, so that the total variance in time Δt is $\sigma^2 \Delta t$.

We model the evolution of the stock price over an instantaneous time Δt by means of a one-period binomial tree. The expected drift and expected volatility (quantities that determine the *future* evolution, for it is the future we are concerned with) must be extracted or predicted from what we observe about the stock price from an investor's point of view. We have to calibrate the binomial evolution so as to be consistent with Equation 6.1, which means determining the parameters *q, u,* and *d*. The parameter *q* is the investor's estimates of the future probability of a move up with logarithmic return *u*. The investor's point of view is often called the *q measure.*



How do we choose *q*, *u* and *d* to match the continuous-time evolution of Equation 6.1? To match the mean and variance of the return, must require that

$$\begin{aligned} qu + (1-q)d &= \mu\Delta t \\ q[u-\mu\Delta t]^2 + (1-q)[d-\mu\Delta t]^2 &= \sigma^2\Delta t \end{aligned} \qquad \text{Eq.6.2}$$

By substituting the first equation for $\mu\Delta t$ into the second, one can rewrite the two equations above as

$$\begin{aligned} qu + (1-q)d &= \mu\Delta t \\ q(1-q)(u-d)^2 &= \sigma^2\Delta t \end{aligned} \qquad \text{Eq.6.3}$$

There are two constraints on the three variables *q, u,* and *d,* so there are a variety of solutions to the equation, and we have the freedom to pick convenient

ones. Convenience here means "easy to think about" or "converges faster to the continuous limit."

6.1.1 First Solution: The Cox-Ross-Rubinstein Convention

Choose $u + d = 0$ for convenience, so that stock price always returns to the same level after successive up and down moves, thereby keeping the center of the tree fixed. Then

$$(2q-1)u = \mu\Delta t$$
$$4q(1-q)u^2 = \sigma^2\Delta t$$

Notice that q = 1/2 if Δt = 0 so write q ~ $\frac{1}{2} + \varepsilon$. Then squaring the first equation and dividing by the second leads to

$$\frac{(2q-1)^2}{4q(1-q)} \approx 4\varepsilon^2 = \frac{\mu^2\Delta t}{\sigma^2}$$

so that

$$\varepsilon \approx \frac{\mu}{2\sigma}\sqrt{\Delta t}$$
$$q \approx \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t} \qquad \text{Eq.6.4}$$
$$u = \sigma\sqrt{\Delta t}$$
$$d = -\sigma\sqrt{\Delta t}$$

We can check that these choices lead to the right drift and volatility. The mean return of the binomial process is

$$\left(\frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t}\right)(\sigma\sqrt{\Delta t}) - \left(\frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta t}\right)(\sigma\sqrt{\Delta t}) = \mu\Delta t$$

The variance is

$$q(1-q)(u-d)^2 \approx \frac{1}{4}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\left(1 - \frac{\mu}{\sigma}\sqrt{\Delta t}\right)4\sigma^2\Delta t \approx \sigma^2\Delta t - \mu^2(\Delta t)^2$$

This variance is a little smaller than it should be, because of the (Δt^2) term. But as $\Delta t \to 0$, this term becomes negligible relative to the $O(\Delta t)$ term, so

that the convergence to the continuum limit is a little slower than if it matched the variance exactly.

For small enough Δt there is no riskless arbitrage with this convention – the up return $\sigma\sqrt{\Delta t}$ in the binomial tree always lies above the return $\mu\Delta t$, which lies above the down state $-\sigma\sqrt{\Delta t}$, because $(\Delta t)^{0.5} >> \Delta t$.

6.1.2 Another Solution: The Jarrow-Rudd Convention

We must satisfy the constraints

$$\begin{aligned} qu + (1-q)d &= \mu\Delta t \\ q(1-q)(u-d)^2 &= \sigma^2\Delta t \end{aligned}$$

Now for convenience we choose $q = 1/2$, so that the up and down moves have equal probability. Then

$$\begin{aligned} u + d &= 2\mu\Delta t \\ u - d &= 2\sigma\sqrt{\Delta t} \end{aligned}$$

and so

$$\begin{aligned} u &= \mu\Delta t + \sigma\sqrt{\Delta t} \\ d &= \mu\Delta t - \sigma\sqrt{\Delta t} \end{aligned} \qquad \text{Eq.6.5}$$

The mean return is exactly μ; the volatility of returns is exactly σ, so that convergence to the continuum limit is faster than in the Cox-Ross-Rubinstein convention.

Let's look at the evolution of the stock price as we iterate over many time periods; (We'll examine it more closely when we discuss binomialization or discretization of various stochastic processes later.)

$$\begin{aligned} E[S] &= \frac{(e^u + e^d)}{2} S \\ &= e^{\mu\Delta t}\frac{(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})}{2} \approx e^{\mu\Delta t}\left(1 + \frac{\sigma^2\Delta t}{2}\right) \approx e^{\left(\mu + \frac{\sigma^2}{2}\right)\Delta t} \end{aligned}$$

so that the expected return on the stock price is $\mu + \sigma^2/2$.

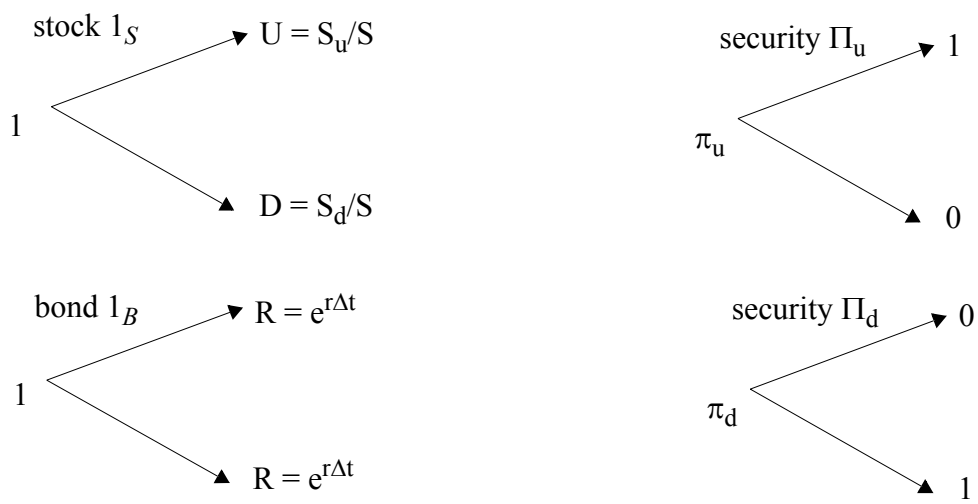
In the limit $\Delta t \to 0$, both the CRR and the JR convention describe the same process, and there are many other choices of u, d, and q that do so too.

Here we are modeling purely geometric Brownian motion which leads to the Black-Scholes formula. We will use these binomial processes, and trinomial generalizations of them, as a basis for modeling more general stochastic processes that can perhaps explain the smile.

6.2 The Binomial Model for Options Valuation

6.2.1 Options Valuation

One can decompose the stock S and the bond B into two primitive state-contingent (Arrow-Debreu) securities Π_u and Π_d that pay out only in the up or down state.



Define $\Pi_u = \alpha 1_S + \beta 1$. Note that because it is riskless, the sum

$$\Pi_u + \Pi_d = 1/R$$

Then

$$\begin{aligned} \alpha U + \beta R &= 1 \\ \alpha D + \beta R &= 0 \end{aligned} \quad \text{so that} \quad \begin{aligned} \alpha &= \frac{1}{(U-D)} \\ \beta &= \frac{-D}{R(U-D)} \end{aligned} \qquad \text{Eq.6.6}$$

and so the securities are given by the linear combinations

$$\Pi_u = \frac{R1_S - D1_B}{R(U-D)} \qquad \Pi_d = \frac{U1_B - R1_S}{R(U-D)} \qquad \text{Eq.6.7}$$

The values of these state-contingent securities are

$$\pi_u = \frac{R-D}{R(U-D)} \equiv \frac{p}{R} \qquad \pi_d = \frac{U-R}{R(U-D)} \equiv \frac{1-p}{R} \qquad \text{Eq.6.8}$$

where

$$p = \frac{R-D}{U-D} \qquad 1-p = \frac{U-R}{U-D} \qquad \text{Eq.6.9}$$

are the risk-neutral no-arbitrage probabilities that don't depend on expected returns at all. This the *p measure*.

Note that the first equation in Equation 6.9 can be rewritten as $pU + (1-p)D = R$, or

$$S = \frac{pS_u + (1-p)S_d}{R} \qquad \text{Eq.6.10}$$

so that in this measure the current stock price is the risklessly discounted expected future value, or the expected future stock price is the forward price.

Now any option C which pays C_u in the up-state and C_d in the down-state is replicated by $C = C_u\Pi_u + C_d\Pi_d$ with value

$$C = \frac{pC_u + (1-p)C_d}{R} \qquad \text{Eq.6.11}$$

Equation 6.10 and Equation 6.11 express the value of the underlying stock and the replicated option as the discounted expected value of the terminal payoffs in the risk-neutral probability measure defined by p. One can regard Equation 6.10 as *defining* the measure p given the values of S, S_u and S_d; one can regard Equation 6.11 as specifying the value C in terms of the option pay-offs and the value of p.

6.2.2 The Black-Scholes Partial Differential Equation and the Binomial Model

The Black-Scholes PDE can be obtained by taking the limit of the binomial pricing equation as $\Delta t \to 0$. We'll use the Cox-Ross-Rubinstein choice of q, u & d to illustrate this convergence. Let

$$u = \sigma\sqrt{\Delta t} \qquad d = -\sigma\sqrt{\Delta t}$$

Then the option value is given by

$$RC = pC_u + (1-p)C_d \qquad \text{Eq.6.12}$$

where

$$p = \frac{RS - S_d}{S_u - S_d} \qquad 1 - p = \frac{S_u - RS}{S_u - S_d} \qquad \text{Eq.6.13}$$

Now substitute $S_u = e^u S$, $S_d = e^d S$ and $R = e^{r\Delta t}$ in the two equation directly above, so that all terms are re-expressed in terms of the variables r, σ and S.

When you write Equation Eq.6.12 on page 7 in terms of these variables, you obtain

$$e^{r\Delta t} C = pC(e^{\sigma\sqrt{\Delta t}} S, t + \Delta t) + (1 - p)C(e^{-\sigma\sqrt{\Delta t}} S, t + \Delta t)$$

Substituting the equation for p in terms of the same variables, and performing a Taylor expansion to leading order in Δt, one can show that

$$Cr\Delta t = \frac{\partial C}{\partial S}\{rS\Delta t\} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\{S^2\sigma^2\Delta t\} + \frac{\partial C}{\partial t}\Delta t \qquad \text{Eq.6.14}$$

Dividing by Δt leads to the BS equation. Note that the expected growth rate of the stock, μ, appears nowhere in the equation.

You can derive many of the PDEs of stochastic processes (the mean hitting time, for example) in this way.

6.3 Extending the Black-Scholes Model (Read this section but it won't be covered in class.)

Many of the extensions to Black-Scholes involve extending the BS formula by clever transformations of the numeraire in which the stock is valued or the number of shares or the scale in which one measures time.We can start with the simplest case, zero rates and zero dividend yield, and work our way progressively up to more complex cases.

6.3.1 Base Case: Black-Scholes with zero dividend yield, zero rates, and the riskless bond as the numeraire.

$$
\begin{aligned}
C_{BS}(S, t, K, T, \sigma) &= SN(d_1) - KN(d_2) \\
d_{1,2} &= \frac{\ln S/K \pm v^2/2}{v} \\
v &= \sigma\sqrt{T-t}
\end{aligned}
$$

This is really an option to exchange a single bond B with face K for a single stock S. It's more insightful to avoid using prices in dollars, as above, and instead write this using the bond price B as the currency or numeraire ito denominate all prices.

Let $C_B = C/B$ be the Black-Scholes option price in units of B, and let $S_B = S/B$ be the stock price in units of B. Then

$$
\begin{aligned}
C_B &= F(x, v) \\
F(x, \nu) &= xN(d_1) - N(d_2) \\
d_{1,2} &= \frac{\ln x \pm v^2/2}{v} \\
v &= \sigma\sqrt{T-t}
\end{aligned}
\qquad \text{Eq.6.15}
$$

where $x = S_B$.

C_B represents the price of an option on the stock S_B with strike 1_B in units of B. All prices are now dimensionless in terms of dollars.

6.3.2 Moving to non-zero rates

When the interest rate on the bond B is non-zero, the bond B grows at the riskless rate so that $dB = rBdt$. If we denominate all securities in units of B,

then $B_B = 1$ B earns zero interest, and in these units the evolution is analogous to that of Section 6.3.1. We denote the stock price in these units by $S_B = S/B = (S/K)e^{rt}$.

In B units,

$$C_B = F(x, \nu) \qquad \text{Eq.6.16}$$

as in Equation 6.15, except that now $x = S_B = (Se^{r\tau})/K$.

Converting Equation 6.16 into dollars by multiplying both sides by the initial value of B, we obtain for the price C in dollars

$$\begin{aligned} C \equiv BC_{B_M} \equiv Ke^{-r\tau}F(x, \nu) &= Ke^{-r\tau}[xN(d_1) - N(d_2)] \\ &= Ke^{-r\tau}[(Se^{r\tau}/K)N(d_1) - N(d_2)] \\ &= SN(d_1) - Ke^{-r\tau}N(d_2) \end{aligned}$$

which is the standard Black-Scholes formula.

6.3.3 Stochastic interest rates

In the case above, the volatility in the Black-Scholes formula is actually the volatility of the stock S measured in units of the bond price B. If interest rates are stochastic then B is stochastic too, and all that must be changed in the BS formula is the volatility, so that

$$\sigma_{(S/B)}{}^2 = \sigma_S{}^2 + \sigma_B^2 - 2\rho_{S,B}\sigma_S\sigma_B$$

You can usually ignore the volatility of the bond compared to the volatility of the stock, because interest rates volatilities are smaller than stock volatilities and because bonds have lower duration.

For example, if $B = K\exp(-yT)$ the $\left|\frac{dB}{B}\right| \sim Ty\left|\frac{dy}{y}\right|$ and so $\sigma_B \sim yT\sigma_y$.

For T = 1 year, $\sigma_y \sim 0.1$ and y ~ 0.05, we have $\sigma_B \sim 0.005$ or half a vol point, much smaller than the typical 20% volatility of a stock.

6.3.4 Stock with a continuous known dividend yield *d*

When a stock pays dividends at a rate *d* per unit time, it's similar to a dollar in the bank paying continuous interest *r* in its own currency. Just as one dollar grown into $e^{r\tau}$ dollars, so one share will grow in $e^{d\tau}$ shares of stock.

Therefore, to get the payoff of a European option on one share of stock which pays off $max(S_T - K, 0)$ at expiration *T*, you can buy an option on less than one share today, that is on $e^{-d\tau}$ shares today, whose initial value is $Se^{-d\tau}$.

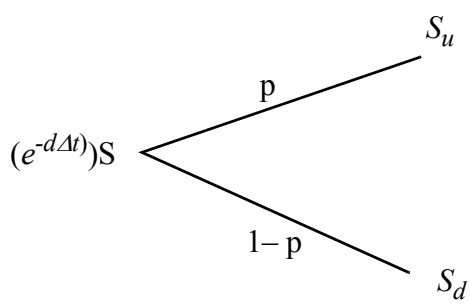
An option on a stock S with dividend yield *d* is therefore equivalent to a Black-Scholes option on a stock whose initial price is $Se^{-d\tau}$. The Black Scholes formula in this case becomes

$$C_{BS}(S, t, K, T, r, d, \sigma) = Se^{-d\tau}N(d_1) - Ke^{-r\tau}N(d_2)$$

$$d_{1,2} = \frac{\ln Se^{(r-d)\tau}/K \pm v^2/2}{v}$$

$$v = \sigma\sqrt{T-t}$$

You can get the same result in the binomial model. If the stock pays a dividend yield *d*, then because one share of stock worth *S* grows to $e^{d\Delta t}$ shares worth S_u or S_d, the tree of *value* (rather than price) is



Then the risk-neutral no-arbitrage growth condition must take account of dividends as well as stock values to define *p* measure, so that

$$pS_u + (1-p)S_d = e^{r\Delta t}(Se^{-d\Delta t}) = Se^{(r-d)\Delta t} \equiv F$$

where *F* is the forward price of the stock, including dividend payments.

Thus

$$p = \frac{F - S_d}{S_u - S_d}$$

Since options pay no dividends, their payoff is discounted at the riskless rate

$$pC_U + (1-p)C_D = Ce^{r\Delta t}$$

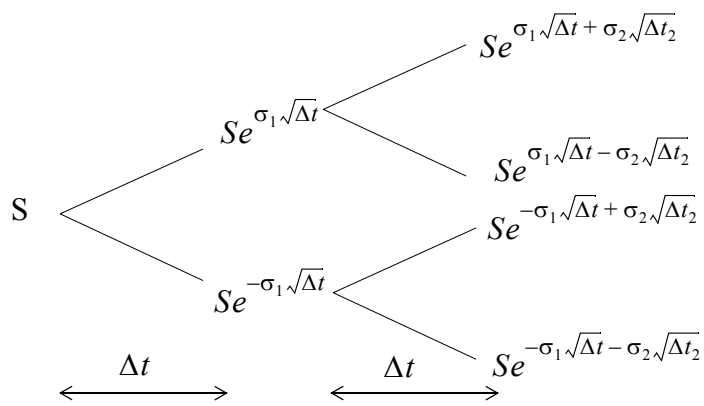
6.4 Extending Black-Scholes for time-dependent deterministic volatility

Black-Scholes and the binomial model assume that σ is constant no matter how *S* and *t* change. Suppose now that the stock volatility σ is a function of (future) time t.

$$\frac{dS}{S} = \mu dt + \sigma(t) dZ$$

How do we modify Black-Scholes or the binomial tree method when there is a term structure of volatilities $\sigma(t)$?

Suppose we try to build a CRR tree with σ_1 in period 1 and σ_2 in period 2.



Then, as you can see, the tree doesn't "close" in the second period unless σ_i is constant. Of course no one can demand that the tree close; it's just computationally convenient in order to avoid an exponentially growing number of final states. But it's preferable to have it close and use the same binomial algorithm for European and American options even when volatility is a deterministic function of time.

To make the tree close, we can instead change the spacing between levels in the tree. Since each move up or down in the price tree from time level *i - 1* to *i* is multiplied $\sigma_i\sqrt{\Delta t_i}$, we can guarantee that the tree will close provided that $\sigma_i\sqrt{\Delta t_i}$ is the same for all periods, or

$$\sigma_1\sqrt{\Delta t_1} = \sigma_2\sqrt{\Delta t_2} = \ldots = \sigma_N\sqrt{\Delta t_N} \qquad \text{Eq.6.17}$$

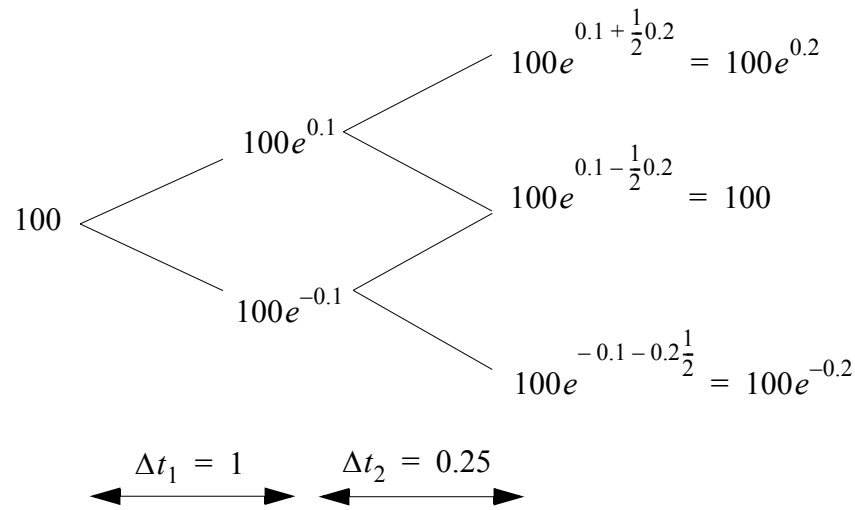
Thus, though the tree looks the same from a topological point of view, each step between levels involves a step in time that is smaller when volatility in the period is larger, and vice versa.

One difficulty (but not an insuperable one) with this approach is that you don't easily know how many time steps you require to get to a definite expiration, because the time steps vary with volatility. Once you know the term structure of volatilities, you can solve for the number of time steps needed.

Here's an illustration on a crude binomial tree with coarse periods. For an accurate calculation we'd need many more periods. Suppose we believe volatility will be 10% in year 1 and 20% in year 2. We choose the first period to be one year long and then solve for the second period.

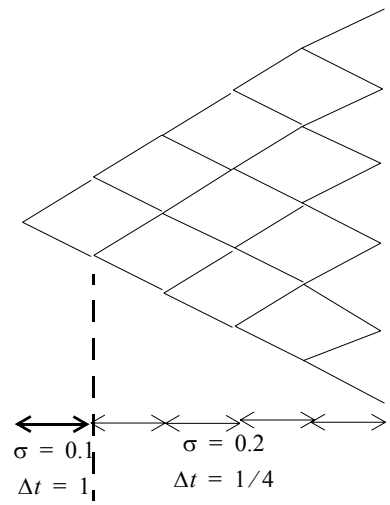
	period 1	period 2
σ	0.1	0.2
σ^2	0.01	0.04
Δt	1	**1/4**

We use the CRR convention in which up and down moves given by $\sigma_i\sqrt{\Delta t_i}$. to illustrate the tree:



In essence, we build a standard binomial tree with price moves generated by $e^{\pm\sigma\sqrt{\Delta t}}$, where $\sigma\sqrt{\Delta t}$ is the same for all periods, and then we choose σ to match the term structure of volatility in each period and then adjust Δt. The stock prices at each node on the tree remains the same as with constant volatility; the tree is topologically identical to a constant volatility tree. However, we reinterpret the times at which the levels occur, and the volatilities that took them there vary according to the table above. A single tree with the same prices at each node can represent different stochastic processes with different volatilities moving through different amounts of time.

The tree in the illustration above extended to 1.25 years. We would need a total of 4 periods to span the entire second year at a volatility of 0.2, but only one period for the first year, so that 5 steps are necessary to span two years.



More generally, if you have a definite time T to expiration, then

$$T = \sum_{i=1}^{N} \Delta t_i = \Delta t_1 \sum_{i=1}^{N} \frac{\sigma_1^2}{\sigma_i^2}$$

and the number of periods necessary to span the time to expiration is given by solving for N in the equation above.

Note 1: Even though the nodes in the tree above have prices corresponding to a CRR tree with $\sigma_i \sqrt{\Delta t_i} = 0.1$, *the binomial no-arbitrage probabilities vary with* Δt_i, because for each fork in the tree,

$$p = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

Even though $e^{\sigma\sqrt{\Delta t}}$ is the same over all time steps Δt, the factor $e^{r\Delta t}$ varies from step to step with the value of Δt, so that p varies from level to level.

Note 2: The total variance at the terminal level of the tree is the same as before

$$\Sigma^2(T-t) \equiv \sum_{i=1}^{N} \sigma_i^2 \Delta t_i \rightarrow \int_t^T \sigma^2(s)ds \approx N\sigma_1^2 \Delta t_1$$

Valuing an option on this tree leads to the Black-Scholes formula with the relevant time to expiration, the relevant interest rates and dividends at each period, and a total variance

$$\Sigma^2 = \frac{1}{T-t}\int_t^T \sigma^2(s)ds \qquad \text{Eq.6.18}$$

Example of a CRR tree with variable volatility, 20% year one, 40% year 2

CRR Tree with variable volatility

Vol is 20% for one year, 40% for second, sqrt of annual variance is 31.6%

sigma		0.2	0.4	0.4	0.4	0.4
delta t		1	0.25	0.25	0.25	0.25
Time	0	1	1.25	1.5	1.75	2
sig*sqrt(delta	0.2	0.2	0.2	0.2	0.2	0.2
u		1.221	1.221	1.221	1.221	1.221
r_annual	0.1					

risk neutral stock tree CRR-style with variable sigma(t)

					271.83
				222.55	
			182.21		**182.21**
		149.18		**149.18**	
	122.14		**122.14**		**122.14**
100.00		**100.00**		**100.00**	
	81.87		**81.87**		**81.87**
		67.03		**67.03**	
			54.88		**54.88**
				44.93	
					36.79

Time	**0**	**1**	**1.25**	**1.5**	**1.75**	**2**
p-tree	**0.699**	**0.510**	**0.510**	**0.510**	**0.510**	

Two year put struck at 100

					0.000
				0.000	
			0.000		0.000
		1.985		0.000	
	6.318		4.149		0.000
9.309		11.139		8.672	
	19.328		18.965		18.127
		28.804		30.613	
			40.465		45.119
				52.713	
					63.212

Constant volatility of 20%

sigma		0.2	0.2	0.2	0.2	0.2
delta t		0.25	0.25	0.25	0.25	0.25
Time	0	0.25	0.5	0.75	1	1.25
sig*sqrt(delta	0.1	0.1	0.1	0.1	0.1	0.1
u		1.105	1.105	1.105	1.105	1.105
r_annual	0.05					

risk neutral stock tree CRR-

						164.87
					149.18	
				134.99		134.99
			122.14		122.14	
		110.52		110.52		110.52
	100.00		100.00		100.00	
		90.48		90.48		90.48
			81.87		81.87	
				74.08		74.08
					67.03	
						60.65

Time	0	0.25	0.5	0.75	1	1.25
					0.536	
				0.536		
			0.536		0.536	
		0.536		0.536		
p-tree	0.536		0.536		0.536	
		0.536		0.536		
			0.536		0.536	
				0.536		
					0.536	
				0.200		
			0.200			
		0.200		0.200		
vol computed	0.200		0.200			
		0.200		0.200		
			0.200			
				0.200		

First 3-months volatility is 10%

sigma		0.1	0.2	0.2	0.2	0.2
delta t		1	0.25	0.25	0.25	0.25
Time	0	1	1.25	1.5	1.75	2
sig*sqrt(delta	0.1	0.1	0.1	0.1	0.1	0.1
u		1.105	1.105	1.105	1.105	1.105
r_annual	0.05					

risk neutral stock tree CRR-style with variable sigma(t)

						164.87
					149.18	
				134.99		134.99
			122.14		122.14	
		110.52		110.52		110.52
	100.00		100.00		100.00	
		90.48		90.48		90.48
			81.87		81.87	
				74.08		74.08
					67.03	
						60.65

Time	0	1	1.25	1.5	1.75	2
					0.536	
				0.536		
			0.536		0.536	
		0.536		0.536		
p-tree	0.725		0.536		0.536	
		0.536		0.536		
			0.536		0.536	
				0.536		
					0.536	
				0.200		
			0.200			
		0.200		0.200		
vol computed	0.089		0.200			
		0.200		0.200		
			0.200			
				0.200		

6.5 Calibrating a binomial tree to term structures

Suppose we know the yield curve and the implied volatility term structure. How do we build a binomial tree to price options that's consistent with it? We have to make sure to use the right forward rate and the right forward volatility at each node.

Example:

Term structure of zero coupons:	**Year 1**	**Year 2**	**Year 3**
	5%	7.47%	9.92%
Forward rates:	5%	$10\% = \frac{(1.0747^2)}{1.05} - 1$	15%
Term structure of Implied vols:	Σ_1	Σ_2	Σ_3
	20%	25.5%	31.1%
Forward vols:	Σ_1	$\Sigma_{12} = \sqrt{2\Sigma_2^2 - \Sigma_1^2}$	$\Sigma_{23} = \sqrt{3\Sigma_3^2 - 2\Sigma_2^2}$
	20%	30%	40%

Now build a (toy) tree with different forward rates/vols:

r:	5%	10%	15%
σ	20%	30%	40%
Δt	Δt_1	$\Delta t_2 = \left(\frac{\sigma_1}{\sigma_2}\right)^2 \Delta t_1$	$\Delta t_3 = \left(\frac{\sigma_1}{\sigma_3}\right)^2 \Delta t_1$
		$= 0.44\Delta t_1$	$0.25\Delta t_1$

A possible scheme:
For the first year use $\Delta t_1 = 0.1$ and take 10 periods of 0.1 years per step.

Then $\Delta t_2 = 0.044$ and we need about 23 periods for the second year.

Finally, $\Delta t_3 = 0.025$ and we need 40 periods for the third year.

In each period the up and down moves in the tree are generated by

$$e^{\sigma\sqrt{\Delta t}} = e^{(0.2)0.316} = 1.065 .$$

Using forward rates and forward volatilities over three years produces a very different tree from using just the three-year rates and volatilities over the whole period, especially for American-style exercise.

6.6 Local volatility binomial models

In the previous section we extended the constant-volatility geometric Brownian motion picture underlying the Black-Scholes model to account for a volatility that can vary with future time. Now we head off in a new direction for several classes -- learning how to make realized volatility $\sigma = \sigma(S, t)$ a function of future stock price S and future time t.

There are several reasons to do this. First, because there is some indication from equity index behavior that realized volatility does go up when the market goes down, at least over short periods; and second, because we want to see if this simple extension of Black-Scholes can then lead to an explanation of the smile.

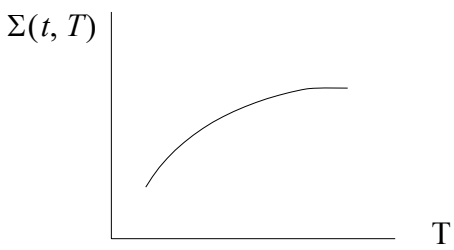
Some references on Local Volatility Models (there are many more).

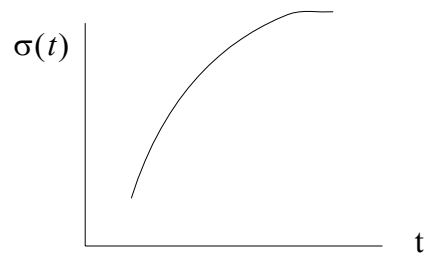
- *The Volatility Smile and Its Implied Tree*, Derman and Kani, RISK, 7-2 Feb.1994, pp. 139-145, pp. 32-39 (see www.ederman.com for a PDF copy of this.
- *The Local Volatility Surface* by Derman, Kani and Zou, Financial Analysts Journal, (July-Aug 1996), pp. 25-36 (see www.ederman.com for a PDF copy of this). Read this to get a general idea of where we're going.
- Rebonato's book, Chapters 11 and 12. Good general coverage.
- Also Clewlow and Strickland's book, *Implementing Options Models*.
- Also Peter James's book *Option Theory.*
- Gatheral's book *The Volatility Surface*.

6.7 Modeling a stock with a variable volatility $\sigma(S,t)$

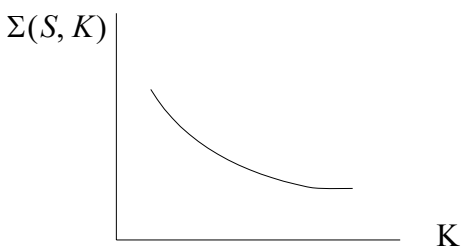
Our aim is to model the evolution of a stock with a variable volatility $\sigma(S, t)$ and then to value options by the principle of no riskless arbitrage. Converting these prices to Black-Scholes implied volatilities, we will then examine the resultant volatility surface $\Sigma(S, t, K, T)$.

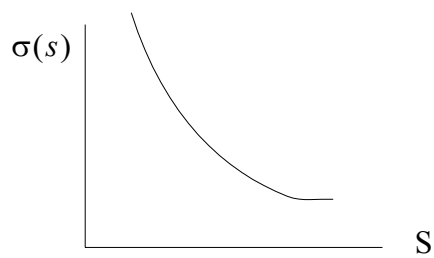
We've just seen that, given a pure term structure of implied volatilities, $\Sigma(t, T)$, we can calibrate the forward volatilities $\sigma(t)$, and that these two quantities are related to each other through Equation 6.18.





Can we expect a similar relationship to hold when we move "sideways" in the strike K and stock-price S direction, relating $\Sigma(S, t, K, T)$ to $\sigma(S, t)$?





More generally, how does the local volatility $\sigma(S, t)$, a function of future stock price S and time t, influence the current implied volatility $\Sigma(S, t, K, T)$ as a function of strike K and expiration T?

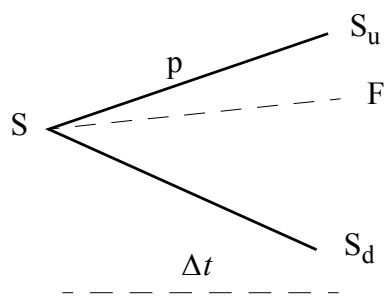
These are some of the questions that will concern us:

- Can we find a unique local volatility surface $\sigma(S, t)$ to match the implied vol surface $\Sigma(S, t, K, T)$?
- Even if we can find the local volatilities that match the implied volatility surface, do they represent what actually goes on in the world?
- What do local volatility models tell us about hedge ratios, exotic values, etc.?

6.8 Binomial Local Volatility Modeling

How do we build a binomial tree that closes (i.e. is not bushy or exponentially growing, in order to avoid computational complexity)?

For any riskless interest rate r and instantaneous volatility $\sigma(S, t)$, the risk-neutral binomial fork for constant spacing Δt looks like this.



S must satisfy the risk-neutral stochastic differential equation

$$\frac{dS}{S} = (r-d)dt + \sigma(S, t)dZ \qquad \text{Eq.6.19}$$

Taking expectations, we deduce that the expected value of S is the forward price $F = Se^{(r-d)\Delta t}$. The binomial version of this equivalence is the expected risk-neutral value one period in the future must satisfy

$$F = pS_u + (1-p)S_d \qquad \text{Eq.6.20}$$

In the case of a discrete dividend D, $F = Se^{r\Delta t} - D$

Furthermore, Equation 6.19 implies that $(dS)^2 = \sigma^2(S, t)S^2 dt$, so that we must require *approximately*, to leading order in Δt, that

$$S^2\sigma^2\Delta t = p(S_u - F)^2 + (1-p)(S_d - F)^2. \qquad \text{Eq.6.21}$$

We can solve for p from Equation 6.19 and then substitute that value into Equation 6.20 to obtain

$$p = \frac{F - S_d}{S_u - S_d}$$
$$(F - S_d)(S_u - F) = S^2\sigma^2\Delta t \qquad \text{Eq.6.22}$$

So, if we know S_d then we can write

$$S_u = F + \frac{S^2\sigma^2\Delta t}{F - S_d} \qquad \text{Eq.6.23}$$

and if we know S_u then we can correspondingly write

$$S_d = F - \frac{S^2\sigma^2\Delta t}{S_u - F}. \qquad \text{Eq.6.24}$$

We now follow the paper *The Volatility Smile and Its Implied Tree*, by Derman and Kani. We can use these formulas to build out the tree at any time level by starting from the middle node and then moving up or down to successive nodes at that level. If we choose the central spine of the tree to be, for example, the CRR central nodes, then, if we know the local volatilities $\sigma(S, t)$ and the forward interest rates at each future period, we can determine the stock prices all the up nodes and down nodes from equations Equation 6.23 and Equation 6.24. Given all the nodes in the tree, we can then use equation for p in Eq.6.22 to compute the risk-neutral probabilities at each node.

There are many ways to choose the central spine of a binomial tree. Here is one:

For every level with an odd number of nodes (1,3,5, etc.) choose the central node to have the initial price S.

For every period with even nodes (2,4,6 etc.) choose the two central nodes in those periods to lie above and below the initial stock price S exactly as in the CRR tree, generated from the previous central node with price S via the up and down factors

$$U = e^{\sigma(S,t)\sqrt{\Delta t}}$$
$$D = e^{-\sigma(S,t)\sqrt{\Delta t}}$$

Here $\sigma(S, t)$ is the local volatility at that stock price S and at the level in the tree corresponding to time t.

We have chosen the spine of the tree to be that of the CRR tree, with all middle nodes having the value S. But you could equally well choose a tree whose spine corresponds to the forward price F of the stock, growing from level to level.

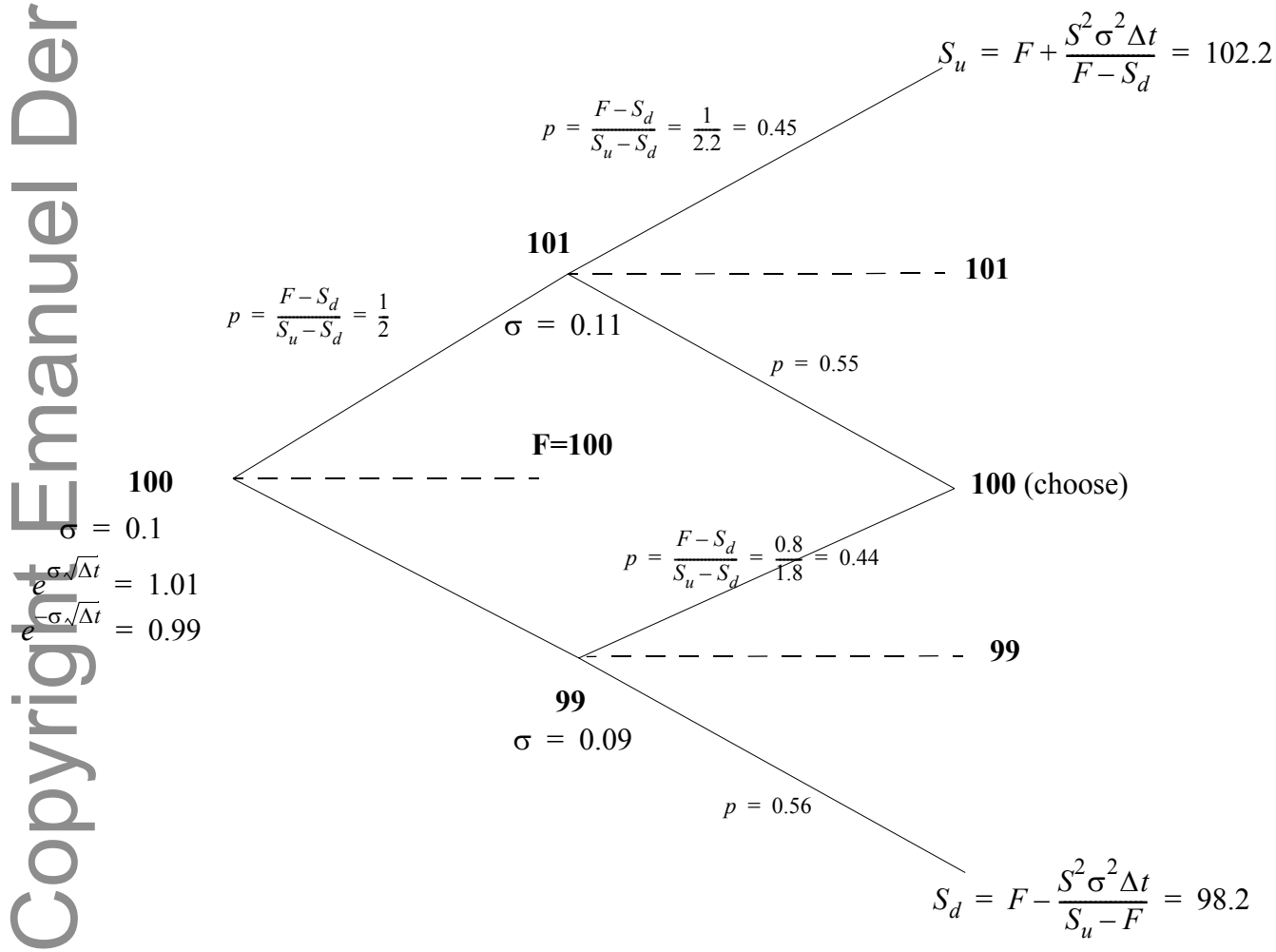
Here's an example with the local volatility a function only of the stock price S:

$S = 100$

$\Delta t = 0.01$; $d = 0$, $r = 0$; $F/S = 1$; $\sqrt{\Delta t} = 0.1$; $e^{\sigma(S)\sqrt{\Delta t}} = e^{\sigma(S)0.1}$ and

$$\sigma(S) = max\left[0.1 + \left(\frac{S}{100} - 1\right), 0\right]$$

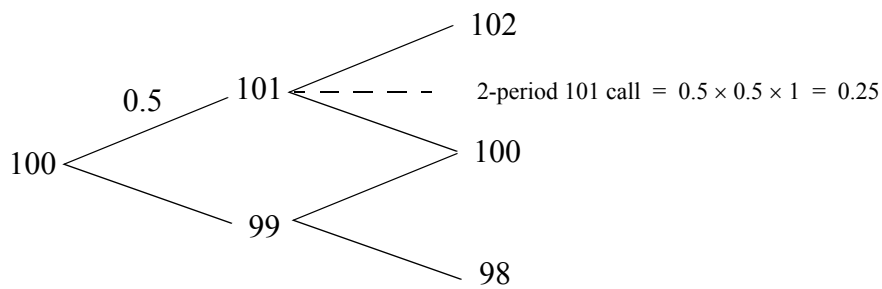
so that local stock volatility starts out at 10% and increases/falls by 1 percentage point for every 1 point rise/drop in the stock price, but never goes below zero. So, for example, $\sigma(100) = 0.1$ and $\sigma(101) = 0.11$



Thus we have a tree that closes, with nodes and probabilities that produce the correct discrete version of the desired diffusion.

Look at the value of a two-period call struck at 101: the payoff at the top node is 1.2 with a risk-neutral probability of (0.5)(0.45) for a value of 0.27.

Let's compare this to the value of a similar call on a CRR tree with a flat 10% volatility everywhere.



You can see that in the local volatility tree, as opposed to the constant volatility tree, there are larger moves up and smaller moves down in the stock price.

Building a binomial tree with variable volatility is in principle possible. In practice, one may get better (i.e. easier to calibrate, more efficient to price with, converging more rapidly as $\Delta t \to 0$,etc.) trees by using trinomial trees or other finite difference PDE approximations. Nevertheless, we will stick to binomial trees in most of our examples here because of the clarity of the intuition they provide.

You can find more references to trinomial trees with variable volatility in Derman, Kani and Chriss, *Implied Trinomial Trees of the Volatility Smile*, The Journal of Derivatives, 3(4) (Summer 1996), pp. 7-22, and also in James' book on Option Theory which is a good general reference on much of this topic.

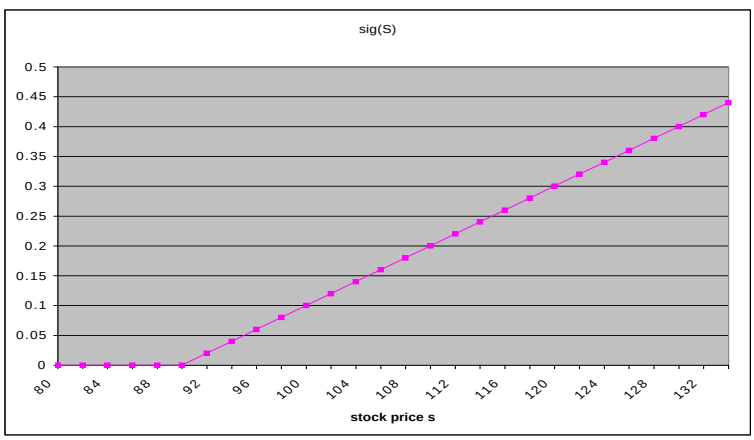
6.9 Looking At The Relation Between Local Volatilities And Implied Volatilities.

Our aim is to build a local volatility tree that matches the smile. What is the relation between local volatilities as a function of S and implied volatilities as a function of K? Here are some examples to illustrate what we might expect and to improve our intuition.

Here is a graph of local volatilities that satisfy a positive skew:

$$\sigma(S) = Max[0.1 + (S/100 - 1), 0].$$

The volatility grows by one point for every one percent rise in the stock price, irrespective of time, but never drops below zero.



Here is the binomial local-volatility tree for the stock price, assuming $\Delta t = 0.01$, $S = 100$, $r = 0$.

This is a tree with flat volatility 0.1, usual CRR type

stock tree **sig = 0.1, p = 0.5**

				104.08
			103.05	
		102.02		102.02
	101.01		101.01	
100		100		100
	99		99	
		98.02		98.02
			97.04	
				96.08

$$\sigma^2 = \frac{(F - S_d)(S_u - F)}{S^2 \Delta t} = 0.01$$

$$\sigma = 0.1$$

stock tree sigma(s,t)

This is a tree with variable local volatility

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

$$\sigma^2 = \frac{(F - S_d)(S_u - F)}{S^2 \Delta t} = \frac{(1.28)(1.53)}{(103.5^2)(0.01)} = 0.018$$

$$\sigma = 0.135$$

p-tree

			0.456
		0.488	
	0.453		0.453
0.498		0.498	
	0.441		0.441
		0.509	
			0.434

vol(stock)

			0.135
		0.122	
	0.110		0.110
0.100		0.100	
	0.090		0.090
		0.082	
			0.074

vol ranges from 13.5 to 7.5 **as stock ranges from 103.5 to 97.4**

The local volatility tree below shows that the CRR implied volatility for a given strike is roughly the average of the local volatilities from spot to that strike. We demonstrate that a call with strike 102 has the same value on the *local volatility tree* as it does on a *fixed-volatility CRR tree* with a volatility of 11%, which is the average of the local volatilities between 100 and 102.

NUMERICAL ILLUSTRATION OF RELATION BETWEEN LOCAL AND IMPLIED VOL

local vol tree

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

stock tree with 11% vol

				104.50
			103.36	
		102.22		102.22
	101.11		101.11	
100.00		100.00		100.00
	98.91		98.91	
		97.82		97.82
			96.75	
				95.70

LOCAL VOL TREE CALL STRUCK AT 102 (sig=12%)

				3.040
			1.510	
		0.790		0.230
	0.386		0.104	
0.204		0.052		0.000
	0.023		0.000	
		0.000		0.000
			0.000	
				0.000

CALL TREE FOR STOCK TREE ON RIGHT STRIKE = 102

				2.498
			1.355	
		0.730		0.224
	0.391		0.112	
0.208		0.055		0.000
	0.028		0.000	
		0.000		0.000
			0.000	
				0.000

Here's another example for the value of a call with strike 103 on the same tree, showing that its implied volatility is about 11.5%, the average of the local volatilities between 100 and 103.

LOCAL VOL TREE CALL STRUCK AT 103 (sig = 13%)

				2.040
			0.929	
		0.453		0.000
	0.205		0.000	
0.102		0.000		0.000
	0.000		0.000	
		0.000		0.000
			0.000	
				0.000

CALL TREE FOR STOCK TREE ON RIGHT STRIKE = 103

				1.707
			0.849	
		0.422		0.000
	0.210		0.000	
0.104		0.000		0.000
	0.000		0.000	
		0.000		0.000
			0.000	
				0.000

local vol tree

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

stock tree with 11.5% vol

				104.71
			103.51	
		102.33		102.33
	101.16		101.16	
100.00		100.00		100.00
	98.86		98.86	
		97.73		97.73
			96.61	
				95.50

6.10 The Rule of 2: Understanding The Relation Between Local and Implied Volatilities

We illustrated above that the implied volatility $\Sigma(S, K)$ of an option is approximately the average of the expected local volatilities $\sigma(S)$ encountered over the life of the option between spot and strike. This is analogous to regarding yields to maturity for zero-coupon bonds as an average over future short-term rates over the life of the bond. In that case, just as forward short-term rates grow twice as fast with future time as yields to maturity grow with time to maturity, so **local volatilities grow approximately twice as fast with stock price as implied volatilities grow with strike**. This relation is the Rule of 2.

Here is a proof in the linear approximation to the skew from the appendix of the paper *The Local Volatility Surface*. Later we'll prove the Rule of 2 more rigorously, but first it's good to understand the intuition behind it.

We restrict ourselves to the simple case in which the value of local volatility for an index is independent of future time, and varies linearly with index level, so that

$$\sigma(S) = \sigma_0 + \beta S \qquad \text{for all time t} \qquad \text{Eq.6.25}$$

If you refer to the variation in future local volatility as the "forward" volatility curve, then you can call this variation with index level the "sideways" volatility curve.

Consider the implied volatility $\Sigma(S,K)$ of a slightly out-of-the-money call option with strike K when the index is at S. Any paths that contribute to the option value must pass through the region between S and K, shown shaded in the figure below. The volatility of these paths during most of their evolution is determined by the local volatility in the shaded region.

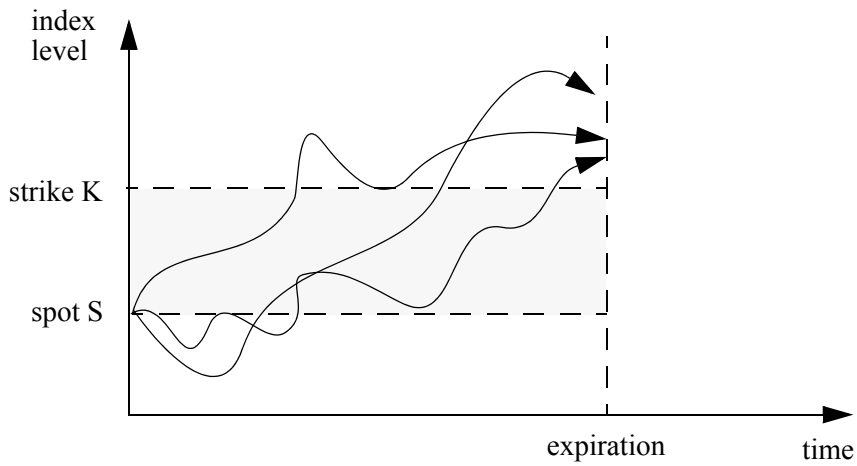
Because of this, you can think of the implied volatility for the option of strike K when the index is at S as the average of the local volatilities over the shaded region, so that

$$\Sigma(S, K) \approx \frac{1}{K-S}\int_S^K \sigma(S')dS' \qquad \text{Eq.6.26}$$

By substituting Eq.6.25 into Eq.6.26 you can show that

$$\Sigma(S, K) \approx \sigma_0 + \frac{\beta}{2}(S+K) \qquad \text{Eq.6.27}$$

FIGURE 6.1. Index evolution paths that finish in the money for a call option with strike K when the index is at S. The shaded region is the volatility domain whose local volatilities contribute most to the value of the call option.



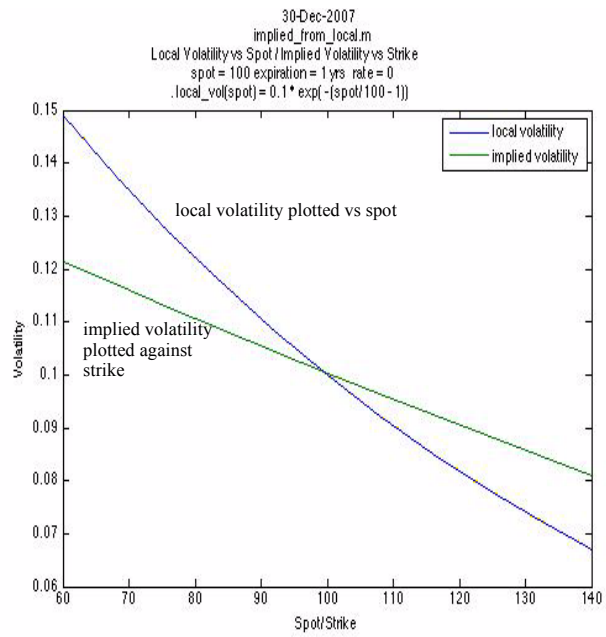
Equation 6.27 shows that, if implied volatility varies linearly with strike K at a fixed market level S, then it also varies linearly at the same rate with the index level S itself. Equation 6.25 then shows that local volatility varies with S at twice that rate. You can also combine Eq.6.25 and Eq.6.27 to write the relationship between implied and local volatility more directly as

$$\Sigma(S, K) \approx \sigma(S) + \frac{\beta}{2}(K - S) \qquad \text{Eq.6.28}$$

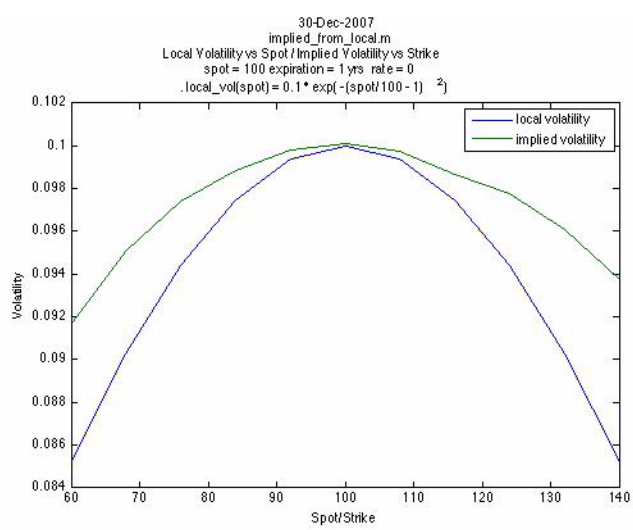
6.11 Some Examples of Local and Implied Volatilities.

Note: In all the figure below, there are two lines or surfaces: the local volatility and the implied volatility. They are plotted against one axis which has the dimension [dollars]. For local volatilities, that axis represent the stock price. For implied volatilities that axis represents the strike of the option. On examination you'll notice that these figures illustrate the Rule of 2.

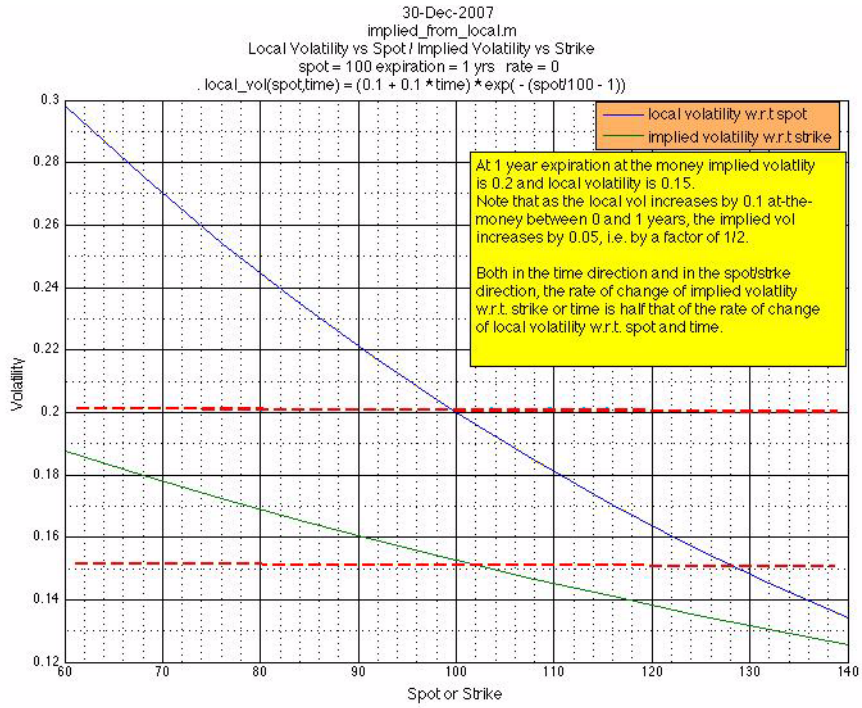
$$\sigma(S,t) = 0.1\exp(-[S/100-1]).$$



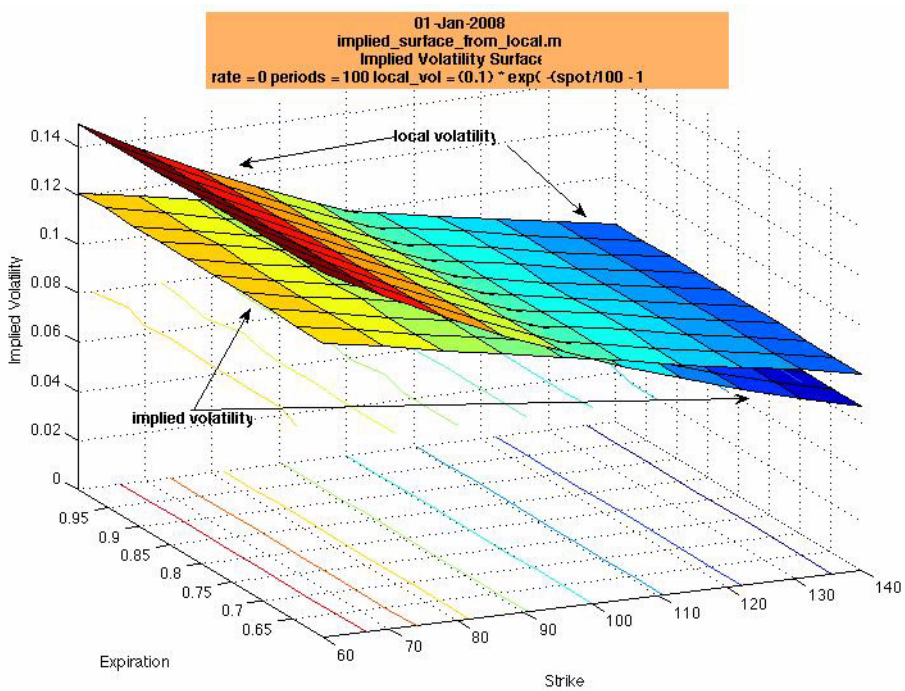
$$\sigma(S,t) = 0.1\exp(-[S/100-1]^2)$$



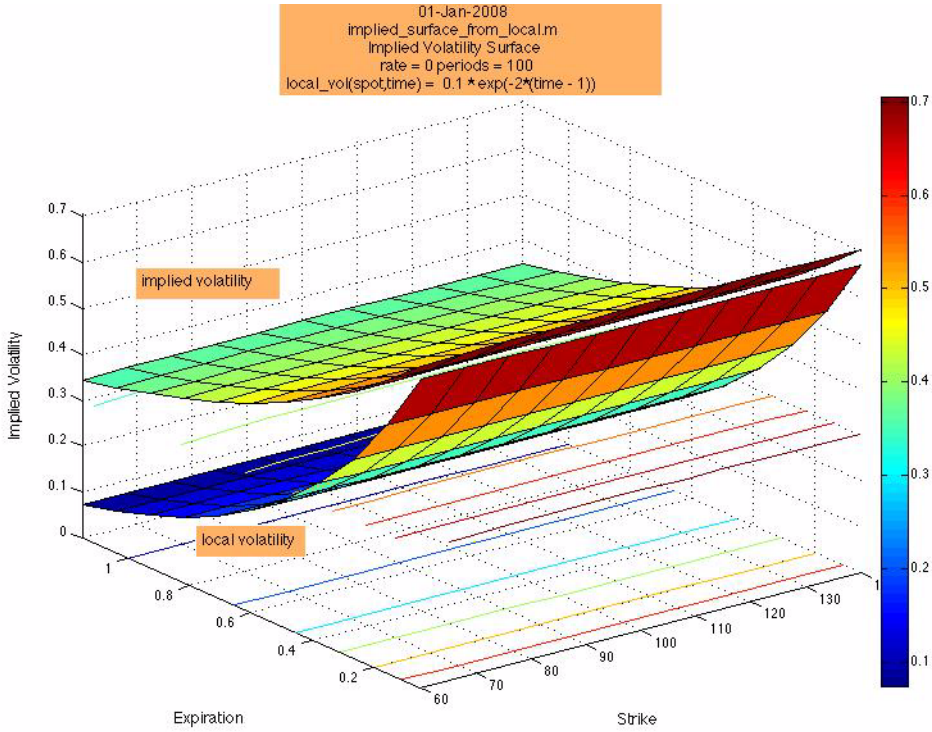
$$1\ \sigma(S, t) = (0.1 + 0.1t)\exp(-[S/100 - 1])$$



Dependent only on S: $\sigma(S, t) = 0.1\exp(-[S/100 - 1])$



Dependent only on t: $\sigma(S, t) = 0.1\exp(-2[t-1])$



Dependent on S and t: $\sigma(S, t) = 0.1\exp(-2[t-1])\exp(-2[S/100-1])$