Lecture 11: Stochastic Volatility Models Cont.

11.1 Valuing Options With Stochastic Volatility

Having understood the qualitative features of stochastic volatility models, we return to examining a full stochastic volatility model. Let's derive a partial differential equation for valuing an option in the presence of stochastic volatility by extending the Black-Scholes riskless-hedging argument.

Assume a general stochastic evolution process for the stock and its volatility are as follows:

$$\begin{aligned} dS &= \mu S dt + \sigma S dW \\ d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\ dWdZ &= \rho dt \end{aligned} \qquad \text{Eq.11.1}$$

The coefficients $p(S, \sigma, t)$ and $q(S, \sigma, t)$ are general functions that can accommodate geometric Brownian motion, mean reversion, or more general behaviors.

Now consider an option that has the value $V(S, \sigma, t)$ and another option $U(S, \sigma, t)$, both dependent on the same stochastic vol and stock price, but with different strikes and/or expirations too.

We can create a portfolio $\Pi = V - \Delta S - \delta U$, short Δ shares of S and short δ options U to hedge V. From Ito's lemma, we have

$$\begin{aligned} d\Pi = {} & \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 dt + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho dt \\ & -\Delta dS \\ & -\delta\left(\frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 dt + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho dt\right) \end{aligned}$$

Collecting the dt, dS and $d\sigma$ terms together we get

$$d\Pi = dt\left[\begin{array}{c} \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right) \end{array}\right]$$

$$+ dS\left(\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta\right) + d\sigma\left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma}\right)$$

We want to create a riskless hedge, and all the risk lies in the dS and $d\sigma$ terms. We can eliminate all the randomness by continuous hedging, choosing Δ and δ to satisfy

$$\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta = 0$$

$$\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma} = 0$$

which gives the hedge ratios

$$\begin{aligned} \Delta &= \frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} \\ \delta &= \frac{\partial V}{\partial \sigma} \Big/ \frac{\partial U}{\partial \sigma} \end{aligned} \qquad \text{Eq.11.2}$$

Then with these hedges in place, the change in value of the hedged portfolio is given by

$$d\Pi = dt\left[\begin{array}{c} \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right) \end{array}\right] \qquad \text{Eq.11.3}$$

Since the increase in the value of Π is deterministic, if there is to be no riskless arbitrage, it must yield the riskless return per unit time, and so

$$d\Pi = r\Pi dt = r[V - \Delta S - \delta U]dt \qquad \text{Eq.11.4}$$

Comparing the last two equations, we have

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho - rV \\ &-\delta\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho - rU\right) \\ &\qquad + r\Delta S = 0 \end{aligned} \qquad \text{Eq.11.5}$$

But we know that $\Delta = \frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S}$ from eliminating the dS risk via Equation 11.2, and so inserting this into Equation 11.5 we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} - rV$$
$$= \delta\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial U}{\partial S} - rU\right)$$

But we also know that $\delta = \frac{\partial V}{\partial \sigma} / \frac{\partial U}{\partial \sigma}$ in order to eliminate the $d\sigma$ risk, and so substituting this into the above equation we obtain

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} - rV}{\frac{\partial V}{\partial \sigma}}$$
$$= \frac{\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial U}{\partial S} - rU}{\frac{\partial U}{\partial \sigma}} \qquad \text{Eq.11.6}$$

Since the LHS is just a function of the option V which we chose arbitrarily, similarly since the RHS is just a function of option U, and since U and V are any securities with completely independent strikes and expirations, each side of Equation 11.6 must be a constant independent of the option parameters, and therefore equal to some function $-\phi(S, \sigma, t)$. This is a separation of variables.

Thus we obtain the valuation equation for an option V:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0 \qquad \text{Eq.11.7}$$

This is the partial differential equation for the value of an option with stochastic volatility σ.

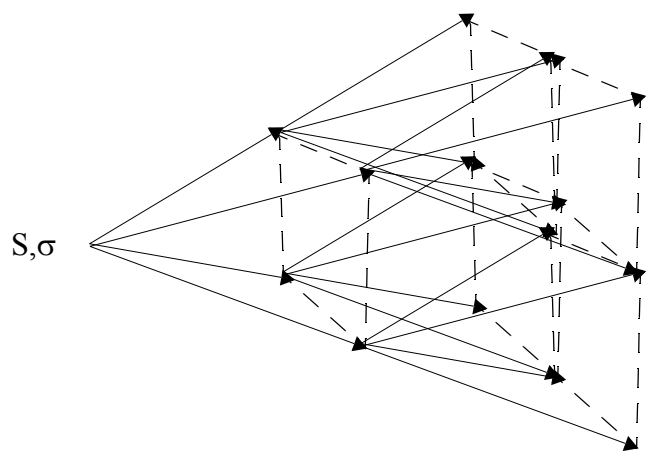
Notice: we don't know the value of the function ϕ!

The meaning of $\phi(S, \sigma, t)$

In Equation 11.7 you can see that ϕ plays the role for stochastic volatility that the riskless rate r plays for a stochastic stock price. In the Black-Scholes case, r is the rate at which the stock price must grow in order that the option price grows at the riskless rate. Similarly, ϕ is the drift that volatility must undergo in order that option prices grow at the riskless rate r. ϕ is not equal to r because σ is not itself a traded security. The stock S and the option V are traded securities, and therefore their risk-neutral drift is r. ϕ is the rate at which volatility σ must grow in order that the price of the option V grows at the rate r when you can hedge away all risk.

From a calibration point of view, ϕ must be chosen to make option prices grow at the riskless rate. If we know the market price of just one option U, and we assume an evolution process for volatility, $d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dZ$, then we can choose/calibrate the effective drift ϕ of volatility so that the value of U obtained from Equation 11.7 matches its market price. Then we can value all other options from the same pde.

In a quadrinomial picture of stock prices where volatility and stock prices are stochastic, as illustrated in the figure below, we much calibrate the drift of volatility ϕ so that the value of an option U is given by the expected risklessly discounted value of its payoffs.



Once we've chosen ϕ to match that one option price, then, assuming we have the correct model for volatility, all other options can be valued risk-neutrally by discounting their expected payoffs.(Of course, it may be naive to assume that just one option can calibrate the entire volatility evolution process.)

Note that even though the payoffs of the option are the same as in the Black-Scholes world, the evolution process of the stock is different, and so the option price will be different too.

11.2 The Characteristic Solution to the Stochastic Volatility Model: The Mixing Formula

You can rewrite the solution to the stochastic volatility equation for an option as the usual discounted risk-neutral expected present value of the payoffs, where the expectation is taken over all future evolution paths of the stock.

$$V = \exp(-r\tau) \sum_{\text{all paths}} p(path) \times \text{payoff}\big|_{path}$$

where $p(path)$ is the risk-neutral probability for that path.

Furthermore, you can characterize each path by its terminal stock price S_T and the average variance along that path. Then one can decompose the sum above into a double sum over all final stock prices and all path volatilities σ_T, where

$$\sigma_T^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt$$

is the time average of the instantaneous variances along the path.

Therefore

$$V = \exp(-r\tau) \sum_{\text{all } \sigma_T} p(\sigma_T, S_T) \left[\sum_{\substack{\text{paths of all } S_T \\ \text{for fixed } \sigma_T}} \text{payoff}\big|_{path} \right]$$

where $p(\sigma_T, S_T)$ is the probability of a particular terminal stock price and path volatility.

If the stock movements are uncorrelated with the volatilities ($\rho = 0$, then the probability $p(\sigma_T, S_T)$ factorizes so that $p(\sigma_T, S_T) = p(\sigma_T)p(S_T|\sigma_T)$, so that

$$V = \sum_{\text{all } \sigma_T} p(\sigma_T) \left[\exp(-r\tau) \sum_{\substack{\text{paths of all } S_T \\ \text{for fixed } \sigma_T}} p(S_T|\sigma_T)\text{payoff}\big|_{path} \right]$$

Now the expected discounted value of the sum of the payoffs over all stock prices contingent on a fixed average path volatility is equal to the Black-Scholes formula, so that

$$V = \sum_{\sigma_T} p(\sigma_T) \times BS(S, K, r, \sigma_T, T)$$

The stochastic volatility solution for zero correlation is the weighted sum over the Black-Scholes solutions for different volatilities. This intuitively pleasing result is often called the "mixing" theorem and was first derived by Hull and White.

For non-zero correlation there are similar formulas, of the form

$$V = E[BS(S'(\sigma_T, \rho), K, r, \sigma_T(\rho), T)] \qquad \text{Eq.11.8}$$

where the stock price argument in the Black-Scholes formula is replaced by a shifted stock price $S'(\sigma_T, \rho)$ and the volatility σ_T is also replaced by a shifted volatility $\overline{\sigma_T}'(\rho)$ that depends upon the correlation ρ, so it's not quite as useful or intuitive.

You can find the elaborations of these results in the book by Fouque, Papanicolaou and Sircar, and in a paper by Roger Lee, *Implied and Local Volatilities under Stochastic Volatility*, International Journal of Theoretical and Applied Finance, 4(1), 45-89 (2001).

11.3 The Smile That Results From Stochastic Volatility

11.3.1 The zero-correlation smile depends on moneyness

The mixing theorem we just proved says that you average Black-Scholes solutions over the volatility distribution to get the stochastic volatility solution. Let's look at a simple case where the path volatility can be one of two values, either high or low, with equal probability, so that

$$C_{SV} = \frac{1}{2}[C_{BS}(S, K, \sigma_H) + C_{BS}(S, K, \sigma_L)] \qquad \text{Eq.11.9}$$

Because the Black-Scholes equation is homogeneous in S and K, we can write

$$C_{SV} = \frac{1}{2}\left[SC_{BS}\left(1, \frac{K}{S}, \sigma_H\right) + SC_{BS}\left(1, \frac{K}{S}, \sigma_L\right)\right] = Sf\left(\frac{K}{S}\right)$$

Now, in terms of Black-Scholes implied volatilities Σ, then

$$C_{SV} = Sf\left(\frac{K}{S}\right) \equiv SC_{BS}\left(1, \frac{K}{S}, \Sigma\right)$$

and so necessarily

$$\Sigma = g\left(\frac{K}{S}\right)$$

Implied volatility is a function of moneyness in stochastic volatility models with zero correlation (conditional on the state of volatility itself not changing).

In that case, deriving Euler's equation, we get

$$\frac{\partial\Sigma}{\partial S} = \left(-\frac{K}{S^2}\right)g'$$

$$\frac{\partial\Sigma}{\partial K} = \frac{1}{S}g'$$

$$S\frac{\partial\Sigma}{\partial S} + K\frac{\partial\Sigma}{\partial K} = 0$$

So, we have a relation between delta and the skew, where the partial derivatives keep the volatility distribution constant.

At the money or close to it,

$$\frac{\partial \Sigma}{\partial S} \approx -\frac{\partial \Sigma}{\partial K}$$

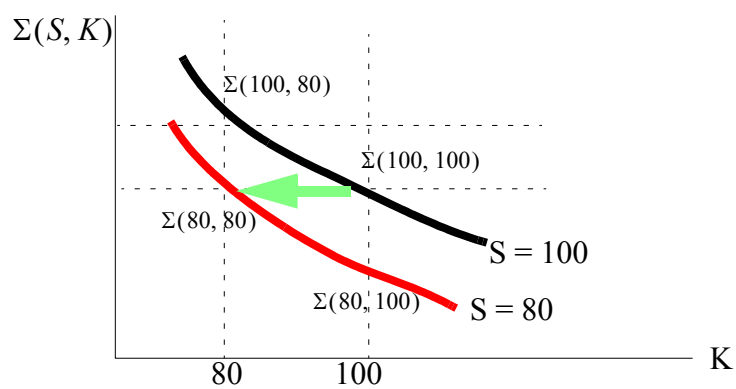
which is just the opposite of what we got with local volatility models.

Approximately, you can think of this equation as saying that implied volatility is a function of $(S - K)$:

$$\Sigma \approx \Sigma(S - K)$$

so that implied volatility is the same whenever $S = K$, i.e. at the money.

In stochastic volatility models, *conditioned on the current volatility remaining the same,* we get the graph



Note that the volatility of all options drops when the stock price drops. Of course if the distribution of volatility changes, then the whole curve can move.

11.3.2 The zero correlation smile is symmetric

The mixing theorem implies that you can write the stochastic volatility call price as an average over the distribution $\phi(\sigma_T)$ of the path volatility σ_T over the life of the option.

$$C_{SV} = \int_0^\infty C_{BS}(\sigma_T)\phi(\sigma_T)d\sigma_T \qquad \text{Eq.11.10}$$

Let's do a Taylor expansion about the average value $\overline{\sigma_T}$ of the path volatility, where, for notational simplicity, we omit the subscript *T*, henceforth denoting σ_T by σ.

$$C_{SV} = \int_0^\infty C_{BS}(\bar{\sigma} + \sigma - \bar{\sigma})\phi(\sigma)d\sigma$$

$$= \int \left\{ C_{BS}(\bar{\sigma}) + \left[\frac{\partial}{\partial\sigma}C_{BS}(\bar{\sigma})\right](\sigma - \bar{\sigma}) + \frac{1}{2}\frac{\partial^2}{\partial\sigma^2}C_{BS}(\bar{\sigma})(\sigma - \bar{\sigma})^2 + \ldots \right\}\phi(\sigma)d\sigma$$

$$= C_{BS}(\bar{\sigma}) + 0 + \frac{1}{2}\left[\frac{\partial^2}{\partial\sigma^2}C_{BS}(\bar{\sigma})\right]var[\sigma] + \ldots$$

where $var[\sigma]$ is the *variance of the path volatility of the stock over the life* τ *of the option*.

Now, using the Black-Scholes equation as our quotation mechanism for options prices, we will write this stochastic volatility solution in terms or the Black-Scholes implied volatility Σ as

$$C_{SV} \equiv C_{BS}(\Sigma) = C_{BS}(\bar{\sigma} + \Sigma - \bar{\sigma})$$

$$= C_{BS}(\bar{\sigma}) + \left[\frac{\partial}{\partial\sigma}C_{BS}(\bar{\sigma})\right](\Sigma - \bar{\sigma}) + \ldots$$

Equating the RHS of the above two expressions for C_{SV}, we can find an expression for Σ in the stochastic volatility model:

$$\Sigma \approx \bar{\sigma} + \frac{\frac{1}{2}\left[\frac{\partial^2}{\partial\sigma^2}C_{BS}(\bar{\sigma})\right]var[\sigma]}{\frac{\partial}{\partial\sigma}C_{BS}(\bar{\sigma})} \qquad \text{Eq.11.11}$$

We can use our knowledge of the derivatives of the Black-Scholes formula to find the functional form of $\Sigma(S, K)$.

Write the Black-Scholes solution for a call with time-averaged volatility σ as

$$C_{BS} = \exp(-r\tau)[S_F N(x) - KN[x - v]] \qquad \text{Eq.11.12}$$

where

$$x = \frac{\ln S_F/K + v^2/2}{v} \qquad \text{Eq.11.13}$$
$$v = \sigma\sqrt{\tau}$$

Now we need to evaluate the derivatives in Equation 11.11. First we find the derivatives of $N(x)$:

$$x = \frac{\ln S_F/K + v^2/2}{v} \qquad \frac{\partial x}{\partial v} = \frac{-\ln S_F/K + v^2/2}{v^2}$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(\frac{-y^2}{2}\right)dy$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) \qquad N'(x-v) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\exp\left(vx - \frac{v^2}{2}\right) = \frac{S_F}{K}N'(x)$$

$$N''(x) = -xN'(x)$$

Now we can evaluate the derivatives of C_{BS} in Equation 11.11:

$$e^{r\tau}C_{BS} = S_F N(x) - KN(x-v)$$

$$e^{r\tau}\frac{\partial C_{BS}}{\partial v} = S_F N'(x)\frac{\partial x}{\partial v} - KN'(x-v)\left(\frac{\partial x}{\partial v} - 1\right) = S_F N'(x)$$

$$e^{r\tau}\frac{\partial^2 C_{BS}}{\partial v^2} = S_F N''(x)\frac{\partial x}{\partial v} = -x\frac{\partial x}{\partial v}S_F N'(x)$$

Therefore, in Equation 11.11

$$\begin{aligned}
\frac{\partial^2 C_{BS}}{\partial \sigma^2}\Big/\frac{\partial C_{BS}}{\partial \sigma} &= \sqrt{\tau}\left[\frac{\partial^2 C_{BS}}{\partial v^2}\Big/\frac{\partial C_{BS}}{\partial v}\right] \\
&= -\sqrt{\tau}x\frac{\partial x}{\partial v} \\
&= -\sqrt{\tau}\left[\frac{\ln S_F/K + v^2/2}{v}\right]\left[\frac{-\ln S_F/K + v^2/2}{v^2}\right] \qquad \text{Eq.11.14} \\
&= \sqrt{\tau}\left[\frac{(\ln S_F/K)^2 - v^4/4}{v^3}\right]
\end{aligned}$$

Then we can convert Equation 11.11 into

$$\Sigma \approx \bar{\sigma} + \frac{1}{2} var[\sigma]\sqrt{\tau}\left[\frac{(\ln S_F/K)^2 - \bar{v}^4/4}{\bar{v}^3}\right] \qquad \text{Eq.11.15}$$

where $\bar{v} = \bar{\sigma}\sqrt{\tau}$, and note that $\bar{\sigma}$ is the cross sectional average of the path volatility over the life of the option

The RHS of Equation 11.15 is a quadratic function of $\ln\frac{S_F}{K}$ and therefore produces a parabolically shaped smile that varies as $(\ln S_F/K)^2$ or $(K - S_F{}^2)$ as you move away from the forward price. It is a sticky moneyness smile, a function of K/S_F alone, with no absolute scale as occurs in local volatility models.

You can use the definition $v = \bar{\sigma}\sqrt{\tau}$ to rewrite Equation 11.15 and so obtain the following expression for implied volatility in an uncorrelated stochastic volatility model

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2} var[\sigma]\left[\frac{(\ln S_F/K)^2 - (\bar{\sigma}^4\tau^2)/4}{\bar{\sigma}^3\tau}\right] \qquad \text{Eq.11.16}$$

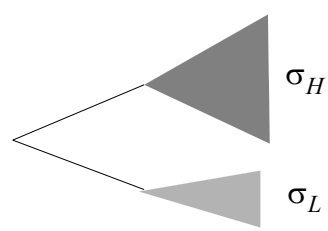
where, we stress again, $var[\sigma]$ is the *variance of the path volatility of the stock over the life of the option*.

11.3.3 A Simple Two-State Stochastic Volatility Model

For simplicity, let's start with a simple two-state stochastic volatility model for quantitatively challenged people. Assume that the stock spends half its time in a high-volatility state σ_H and the other half in a low-volatility state σ_L.

! Then, according to the mixing theorem.

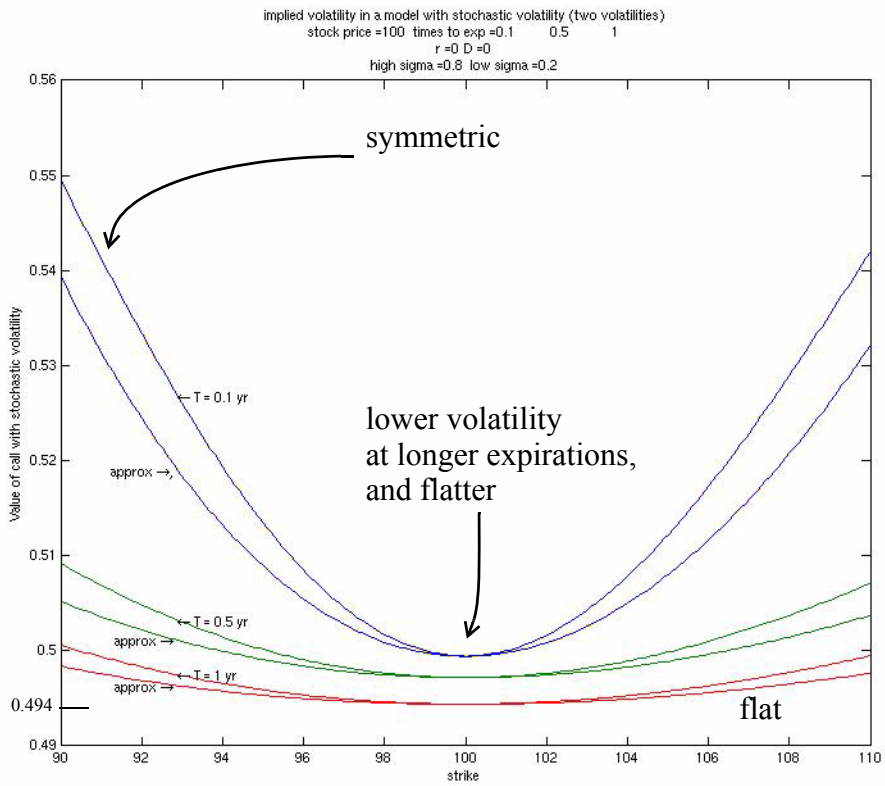
$$C_{SV} = \frac{1}{2}[C_{BS}(S, K, \sigma_H) + C_{BS}(S, K, \sigma_L)]$$



Let the low volatility be 20% (i.e. 0.2) and the high volatility 80% (i.e. 0.8) with a mean volatility of 50% (i.e. 0.5), so that the variance of the volatility is $0.5(0.8 - 0.5)^2 + 0.5(0.5 - 0.2)^2 = 0.09$ per year. In the figure below we

show the smile corresponding to the exact mixing formula, Equation 11.9, together with the approximation of Equation 11.16.

The Volatility Smile in a Two-Volatility Model With Zero Correlation



Notice that

1. the smile with zero correlation is symmetric;
2. the long-expiration smile is relatively flat, while the short expiration skew is more curved (note the τ^{-1} coefficient of $(\ln S/K)^2$ in Equation 11.16); and
3. at the forward price of the stock, the at-the-money implied volatility decreases monotonically with time to expiration, and lies below the mean volatility of 0.5, because of the negative convexity of the Black-Scholes options price at the money.

In either case, you can see that the approximate solution of Equation 11.16 works quite well.

At-the-money, with these parameters, Equation Eq.11.16 reduces to

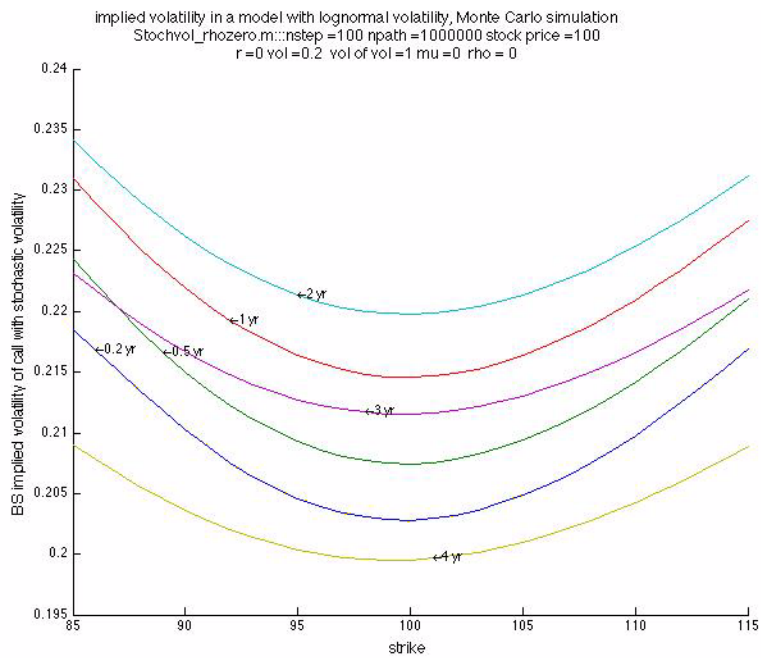
$$\Sigma_{SV}^{ATM} \approx 0.5 - \frac{1}{8}(0.09)\bar{\sigma}\tau \approx 0.5 - 0.0056\tau \qquad \text{Eq.11.17}$$

For $\tau = 1$, Equation 11.17 produces an at-the-money volatility of 0.494, which agrees well with the figure above.

11.3.4 The Smile for GBM Stochastic Volatility

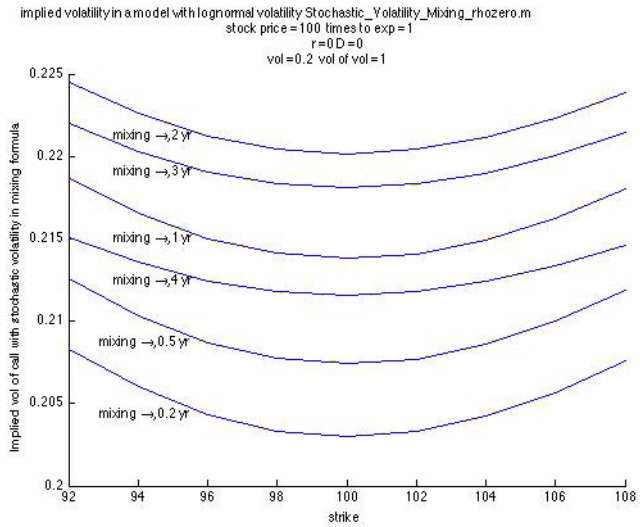
Now, rather than sticking with the simple two-state model, let's look at a more sophisticated continuous distribution of stochastic volatilities undergoing geometric Brownian motion, as the stock price does.

The figure below illustrates the one-year smile for a continuous geometric Brownian diffusion of volatility given by $d\sigma = a\sigma dt + b\sigma dZ$ with zero correlation ρ, an initial volatility of 0.2 and a volatility of volatility of 1.0, calculated by straightforward Monte Carlo simulation of stock paths.



Note the still symmetric smile. The level of at-the-money volatility is now no longer monotonic with time to expiration, but first increases with τ and then decreases. Note also that the skew flattens with increasing τ.

Here is a similar graph for options prices computed via the mixing formula, which is more accurate. The same non-monotonic behavior of at-the-money volatility w.r.t τ is apparent.



You can understand the non-monotonic behavior of at-the-money volatility approximately as follows. From Equation 11.16,

$$\Sigma_{SV}^{ATM} \approx \bar{\sigma} - \frac{1}{8} var[\sigma]\bar{\sigma}\tau \sim \bar{\sigma}\left\{1 - \frac{1}{8} var[\sigma]\tau\right\}$$

For GBM of volatility, the average volatility $\bar{\sigma}$ increases with τ. but the term in parentheses decreases with τ, which can explain the non-monotonic behavior of Σ_{SV}^{ATM}.

You can also (more or less) understand the decreasing curvature of the smile with increasing τ from Equation 11.16, since the term in $(\ln S/K)^2$ is

$$\frac{1}{2} var[\sigma]\left[\frac{(\ln S_F/K)^2}{\bar{\sigma}^3\tau}\right]$$

For GBM, $var[\bar{\sigma}] \sim \tau$ and $\bar{\sigma}$ in the denominator also increases with τ, so that the curvature term tends to decrease as τ increases.

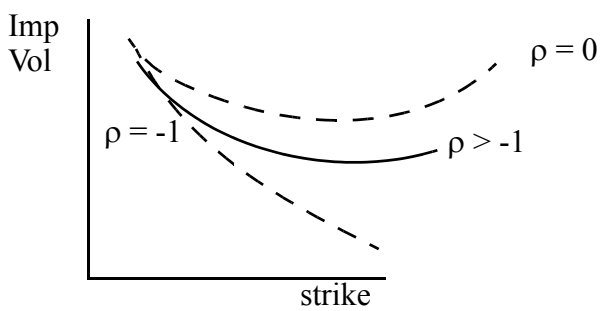
These arguments above are indicative of the trends in the model, but for more details a a better analytic approximation than the simple one we have derived a necessary. These are available in Hull-White and in many of the other books and papers referenced.

11.3.5 Non-zero correlation in stochastic volatility models.

Pure stochastic volatility models with no correlation lead to a symmetric smile. That's not a bad description of currency options markets, which do seem to be driven by changes in volatility, but in general the smile can be asymmetric. To achieve that in a stochastic volatility model you need to add a non-zero correlation.

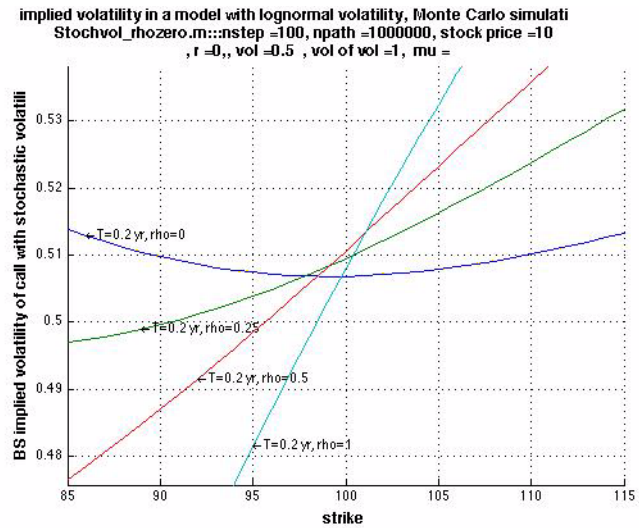
The effect of the correlation between stock price and volatility is to remove the symmetry of the smile that was present in the zero correlation case. The smile still depends on (K/S_F) but the dependence is not quadratic. With a negative correlation, the stock is more likely to move down when volatility goes up, and the skew becomes tilted towards low strikes; with a positive correlation, the reverse is true.

One way to see this is to think about the case when the correlation is -1. Then the stock and its volatility move in tandem and we have a local volatility model with a skew that is purely negative. Now when the correlation moves away from -1 and the local volatility is no longer a pure function of the stock price, a little volatility of volatility adds convexity to the negative skew. We therefore expect the following effect:

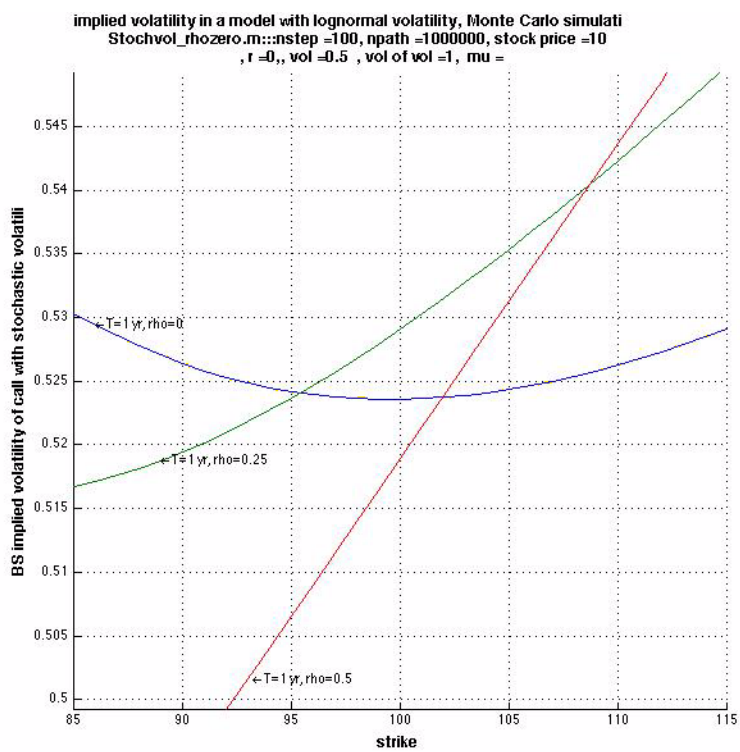


Thus, if you want to account for an S&P-type negative skew with a stochastic volatility model, you need a negative correlation between stock price and volatility. However, to get a very steep short-term skew is difficult in these models; since volatility diffuses continuously in these models, at short expirations volatility cannot have diffused too far. A very high volatility of volatility and very high mean reversion are needed to account for steep short-expiration smiles. (There is more on this in Fouque, Papanicolaou and Sircar's book.)

Here is the result of a Monte Carlo simulation for $\tau = 0.2$ yrs with non-zero ρ. You can see that increasing the value of the correlation steepens the slope of the smile.



Here is a similar figure for $\tau = 1yr$.



11.3.6 The Smile in Mean-Reverting Models

Finally, we explore the smile when volatility mean reverts:

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

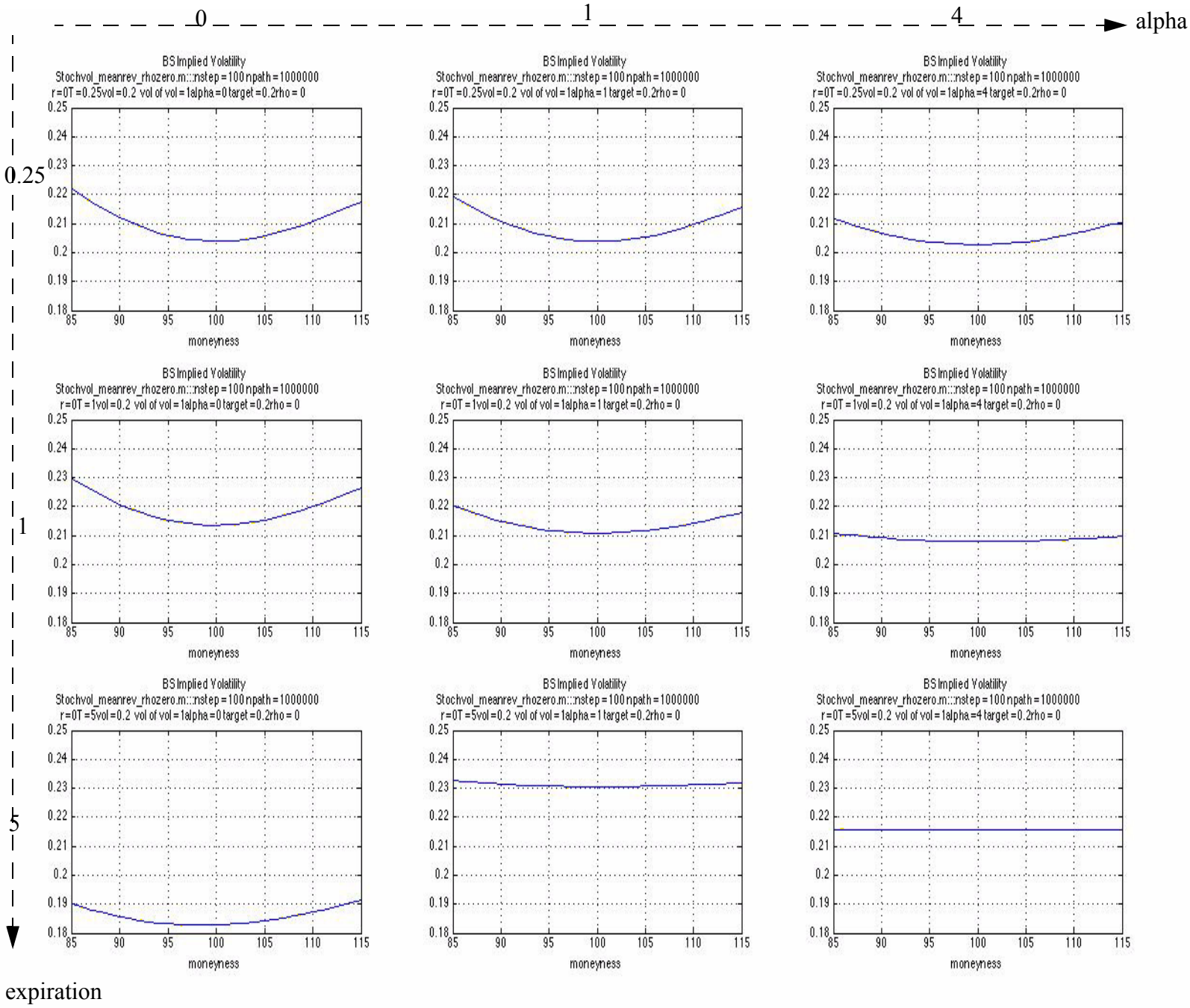
$$d\sigma = \alpha(m - \sigma)dt + \beta\sigma dW$$

$$dZdW = \rho dt$$

The following pages show the results of a Monte Carlo simulation for options prices and the Black-Scholes implied volatility smile.

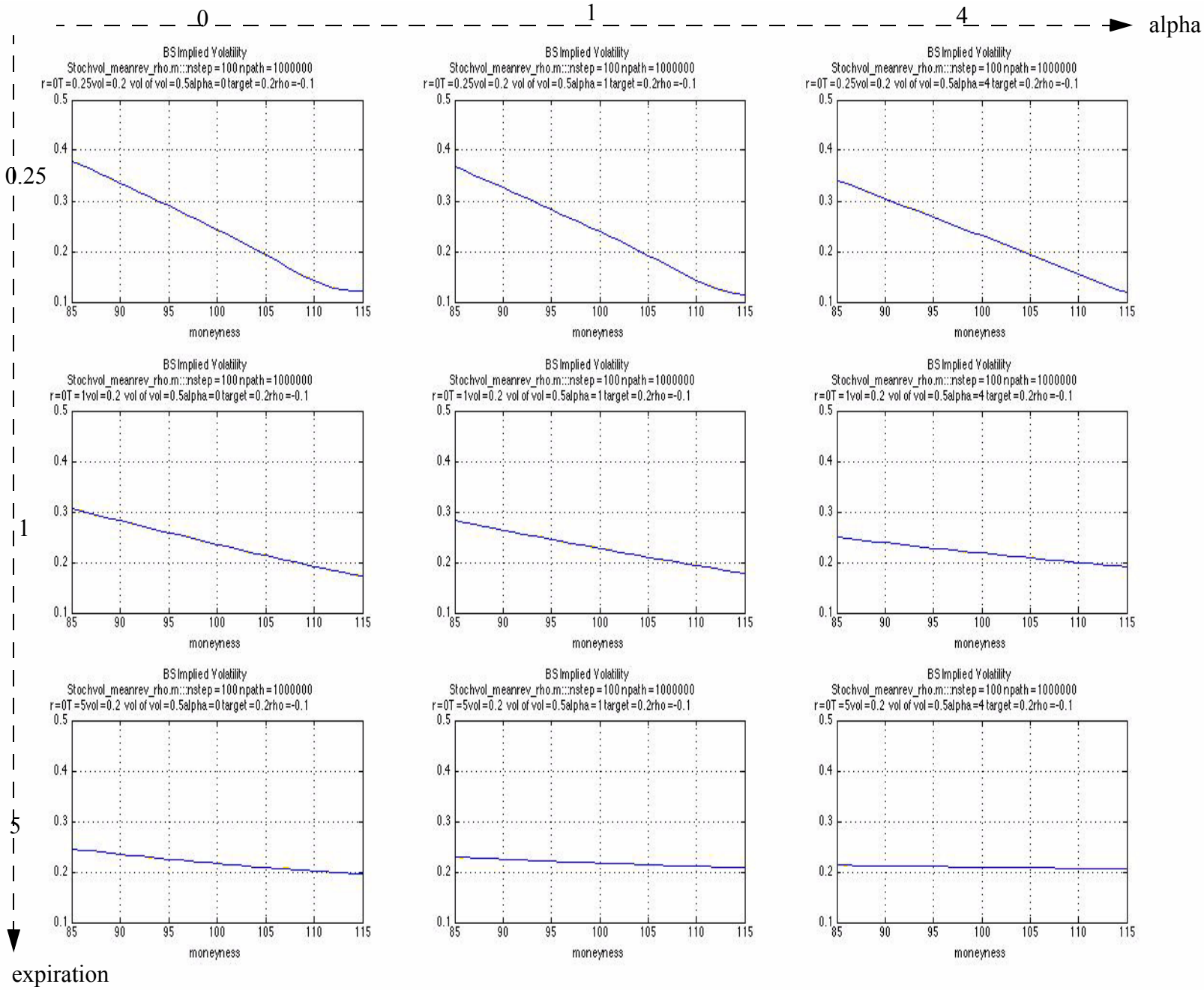
BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation.

Note the flattening of the smile with both expiration and mean-reversion strength α The target volatility and the initial volatility are both 0.2, and the correlation is zero.



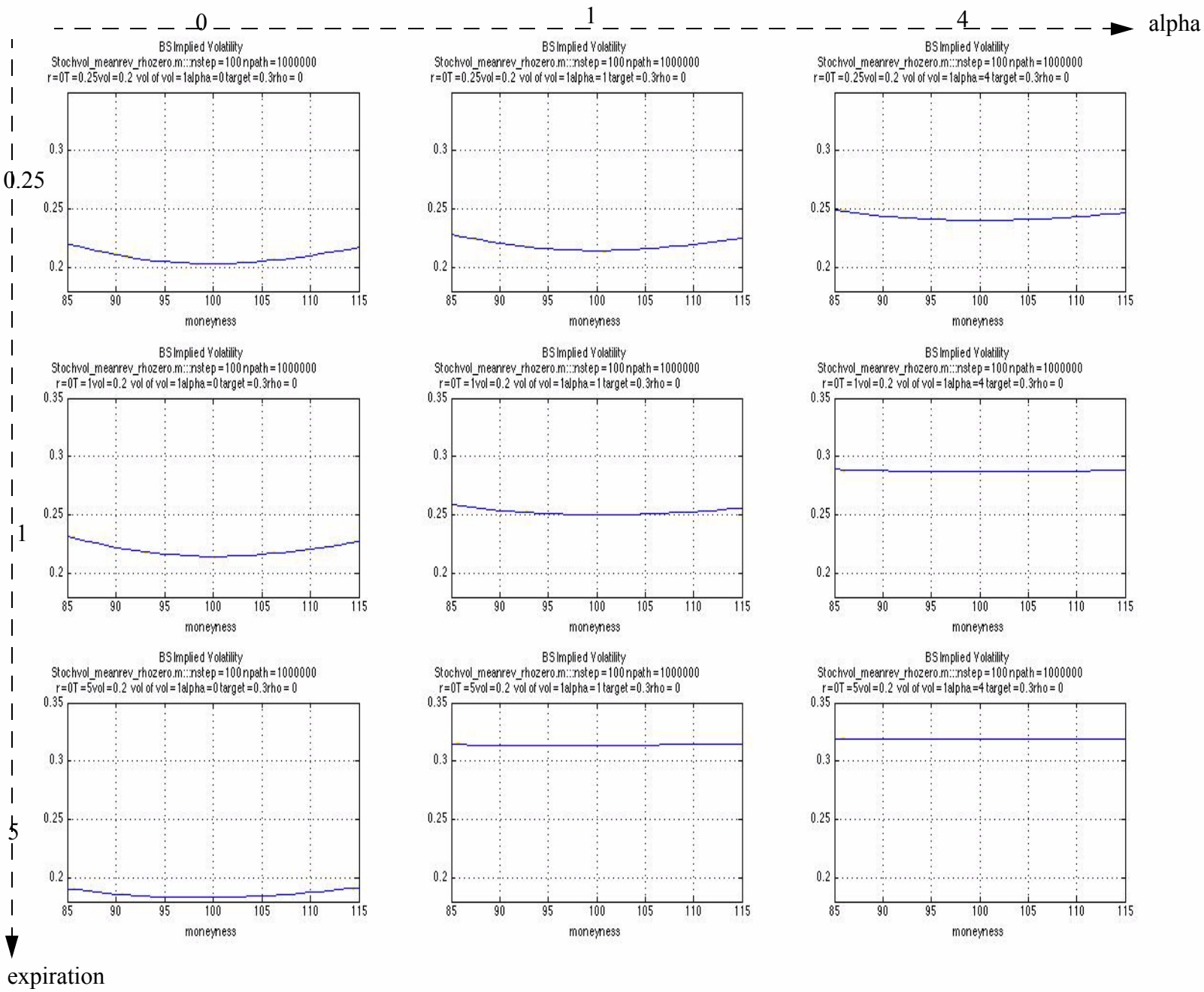
BS Implied Volatility as a function of mean reversion strength and expiration for correlation -0.1.

The negative correlation induces a negative skew. Note the flattening of the smile with both expiration and mean-reversion strength α The target volatility and the initial volatility are both 0.2



BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation, initial volatility 0.2 and target volatility 0.3.

Note the flattening of the smile with both expiration and mean-reversion strength α and the rate of approach of the volatility to the target level.



11.3.7 Mean-Reverting Stochastic Volatility and the Asymptotic Behavior of the Smile.

Volatility tends to be mean reverting, a characteristic we've ignored in the above simulations. The Heston model whose SDE we presented earlier explicitly embodies mean reversion.

We can use the approximation of Equation 11.16, repeated here

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2} var[\sigma]\left[\frac{(\ln S_F/K)^2 - (\bar{\sigma}^4 \tau^2)/4}{\bar{\sigma}^3 \tau}\right], \qquad \text{Eq. 10.30}$$

together with some intuition about the way volatility evolves in various models, to understand how the smile in stochastic volatility models behaves for long and short expirations.

Short Expirations, Zero Correlation

In the limit that $\tau \to 0$, i.e. for very short times to expiration, Equation 11.16 reduces to

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2} var[\sigma]\left[\frac{(\ln S_F/K)^2}{\bar{\sigma}^3 \tau}\right] \qquad \tau \to 0 \text{ limit} \qquad \text{Eq.11.18}$$

Now, if volatility undergoes a diffusion-like geometric Brownian motion, then the variance of volatility grows linearly with time for all times. This is true for times short compared to the mean-reverting parameter α even in the case of mean-reverting stochastic volatility.Thus, in almost any stochastic volatility model, the variance of the instantaneous volatility and of the average volatility over the life of the option, for short expirations τ, increases with τ, and so $var[\bar{\sigma}] = \beta\tau$. Substituting this relation into Equation 11.18 leads to the expression

$$\Sigma_{SV} \approx \bar{\sigma} + \frac{1}{2}\beta\left[\frac{(\ln S_F/K)^2}{\bar{\sigma}^3}\right] \qquad \tau \to 0 \text{ limit} \qquad \text{Eq.11.19}$$

where the τ-dependence has cancelled. Thus, the smile is quadratic and finite as $\tau \to 0$ for short expirations, as illustrated in the figures of Section 11.3.4 (still assuming a zero correlation).

Long Expirations In the limit as $\tau \to \infty$ Equation 11.16 takes the limit

$$\Sigma_{SV} \approx \bar{\sigma} - \frac{1}{2} var[\sigma]\left[\frac{\bar{\sigma}\tau}{4}\right] \qquad \text{Eq.11.20}$$

where $\bar{\sigma}$ is the path volatility over the life of the option

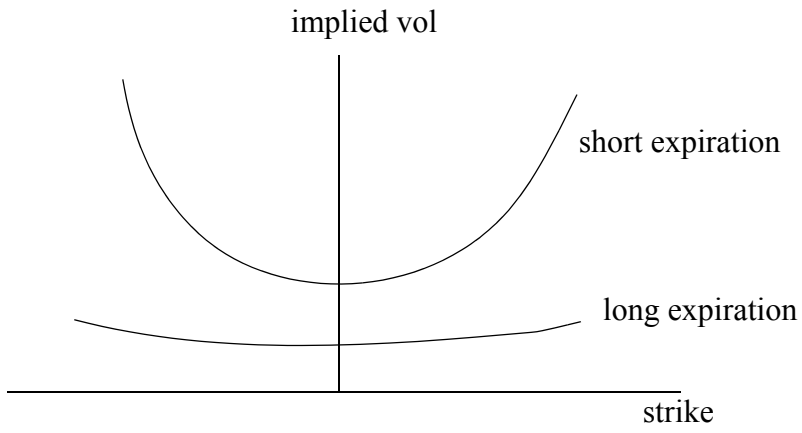
If the instantaneous volatility is mean-reverting, then it cannot get too far away from its mean; for Ornstein-Uhlenbeck processes one can show that the path volatility to expiration $\bar{\sigma}$ converges to a constant along all paths as $\tau \to \infty$, despite the stochastic nature of σ, and therefore $\bar{\sigma}$ has zero variance as $\tau \to \infty$, declining to zero such that $var[\bar{\sigma}] \to \text{const}/\tau$ in this limit. One can understand this qualitatively as follows: if a stochastic variable mean-reverts, its time average along any path settles down to a constant at long times and the variance of the time average approaches zero asymptotically.

Therefore, writing $var[\bar{\sigma}] = (\text{const}/\tau)$ in Equation 11.20, we see that with mean reversion the τ-dependence again vanishes to produce an asymptotic skew that is independent of the time to expiration:

$$\Sigma_{SV} \approx \bar{\sigma} - \frac{\text{const}}{8}\bar{\sigma} \qquad \text{Eq.11.21}$$

The stochastic volatility model with zero correlation yields an implied volatility that is independent of expiration and independent of $\ln\frac{S_F}{K}$ too, meaning *NO smile* at large expirations.Why is the correction term in Equation 11.21 negative? Why does the stochastic volatility lower the implied volatility from the non-stochastic case? You would think, intuitively, that stochastic volatility would result in a higher-than-initial at-the-money implied volatility. This occurs because the option price $C_{BS}(\sigma)$ is not always a convex function of σ. It particular, you can see from Equation 11.14 that the convexity in the price as a function of volatility becomes negative (i.e. the function is concave) at long expirations, and for a concave function $f(x)$, the average of the function $\overline{f(x)}$ is less than the function $f(\bar{x})$ of the average.

Thus, for zero correlation, we expect to see stochastic volatility smiles that look like this:



We can understand this intuitively as follows. In the long run, all paths will have the same volatility if it mean reverts, and so the long-term skew is flat. In the short run, bursts of high volatility act almost like jumps, and induce fat tails

11.4 Comparison of vanilla hedge ratios under Black-Scholes, local volatility and stochastic volatility models when all are calibrated to the same negative skew

Even though all these models can often be calibrated to the same current negative skew for the S&P, they have different evolutions of volatility, different hedge ratios, different deltas, different forward skews.

Black-Scholes: Implied volatility is independent of stock price. The correct delta is the Black-Scholes delta.

Local Volatility: Local volatility goes down as market goes up, so the correct delta is smaller than Black-Scholes.

Stochastic Volatility: Since implied volatility is a function of *K/S*, and since a negative skew means that implied volatility goes up as *K* goes down and *S* stays fixed, then implied volatility must go up as *K* stays fixed and *S* goes up. Therefore, the hedge ratio will be greater than Black-Scholes, contingent on the level of the stochastic volatility remaining the same. But, remember, in a stochastic volatility model there are two hedge ratios, a delta for the stock and another hedge ratio for the volatility, so just knowing how one hedge ratio behaves doesn't tell the whole story anymore.

11.4.1 Best stock-only hedge in a stochastic volatility model

Although stochastic volatility models suggest a hedge ratio greater than Black-Scholes in a negative skew environment, that hedge ratio is only the hedge ratio w.r.t. the stock degree of risk, and doesn't mitigate the volatility risk.

It's interesting to ask what is the best stock-only hedge, best in the sense that you don't hedge the volatility but try to hedge away as much risk as possible with the stock alone. We will show that the best stock-only hedge is a lot like a local volatility hedge ratio, and is indeed smaller than hedge ratio in a Black-Scholes model.

Consider a simplistic stochastic implied volatility model defined by

$$\begin{aligned} \frac{dS}{S} &= \mu dt + \Sigma dZ \\ d\Sigma &= p dt + q dW \\ dZdW &= \rho dt \end{aligned}$$

We have for simplicity assumed that the stock evolves with a realized volatility equal to the implied volatility of the particular option itself. Then for an option $C_{BS}(S, \Sigma)$ where both S and Σ are stochastic, we can find the hedge that minimizes the instantaneous variance of the hedged portfolio. That's as good as we can with stock alone.

This partially hedged portfolio is

$$\pi = C_{BS} - \Delta S$$

long the call and short some number Δ shares of stock. Note that we are **not** hedging the volatility movements, just the stock price movements. Then in the next instant

$$d\pi = \left(\frac{\partial C_{BS}}{\partial S} - \Delta\right) dS + \frac{\partial C_{BS}}{\partial \Sigma} d\Sigma = (\Delta_{BS} - \Delta) dS + \kappa d\Sigma$$

The instantaneous variance of this portfolio is defined by $(d\pi)^2 = var[\pi]dt$ where

$$var[\pi] = (\Delta_{BS} - \Delta)^2 (\Sigma S)^2 + \kappa^2 q^2 + 2(\Delta_{BS} - \Delta)\kappa S \Sigma q \rho$$

The value of Δ that minimizes the residual variance of this portfolio is given by

$$\frac{\partial}{\partial \Delta} var[\pi] = -2(\Delta_{BS} - \Delta)(\Sigma S)^2 - 2\kappa S \Sigma q \rho = 0$$

or

$$\Delta = \Delta_{BS} + \rho\left(\frac{\kappa q}{\Sigma S}\right)$$

The second derivative $\frac{\partial^2}{\partial \Delta} var[\pi]$ is positive, so that this hedge produces a minimum variance.

The hedge ratio Δ is less than Δ_{BS} when ρ is negative, which is the correlation necessary to cause a negative skew. Thus, the best stock-only hedge in a stochastic volatility model tends to resemble the local volatility hedge ratio.

11.5 Conclusion

Stochastic volatility models produce a rich structure of smiles from only a few stochastic variables. There is some element of stochastic volatility in all options markets.SV models provide a good description of currency options markets where the dominant features of the smile are consistent with fluctuations in volatility. However, the stochastic evolution of volatility is not really well understood and involves many at presently unverifiable assumptions.