Goldman Sachs

Quantitative Strategies Research Notes

March 1999

More Than You Ever Wanted To Know* About Volatility Swaps

Kresimir Demeterfi
Emanuel Derman
Michael Kamal
Joseph Zou

* But Less Than Can Be Said

This material is for your private information, and we are not soliciting any action based upon it. This report is not to be construed as an offer to sell or the solicitation of an offer to buy any security in any jurisdiction where such an offer or solicitation would be illegal. Certain transactions, including those involving futures, options and high yield securities, give rise to substantial risk and are not suitable for all investors. Opinions expressed are our present opinions only. The material is based upon information that we consider reliable, but we do not represent that it is accurate or complete, and it should not be relied upon as such. We, our affiliates, or persons involved in the preparation or issuance of this material, may from time to time have long or short positions and buy or sell securities, futures or options identical with or related to those mentioned herein.

This material has been issued by Goldman, Sachs & Co. and/or one of its affiliates and has been approved by Goldman Sachs International, regulated by The Securities and Futures Authority, in connection with its distribution in the United Kingdom and by Goldman Sachs Canada in connection with its distribution in Canada. This material is distributed in Hong Kong by Goldman Sachs (Asia) L.L.C., and in Japan by Goldman Sachs (Japan) Ltd. This material is not for distribution to private customers, as defined by the rules of The Securities and Futures Authority in the United Kingdom, and any investments including any convertible bonds or derivatives mentioned in this material will not be made available by us to any such private customer. Neither Goldman, Sachs & Co. nor its representative in Seoul, Korea is licensed to engage in securities business in the Republic of Korea. Goldman Sachs International or its affiliates may have acted upon or used this research prior to or immediately following its publication. Foreign currency denominated securities are subject to fluctuations in exchange rates that could have an adverse effect on the value or price of or income derived from the investment. Further information on any of the securities mentioned in this material may be obtained upon request and for this purpose persons in Italy should contact Goldman Sachs S.I.M. S.p.A. in Milan, or at its London branch office at 133 Fleet Street, and persons in Hong Kong should contact Goldman Sachs Asia L.L.C. at 3 Garden Road. Unless governing law permits otherwise, you must contact a Goldman Sachs entity in your home jurisdiction if you want to use our services in effecting a transaction in the securities mentioned in this material.

Note: Options are not suitable for all investors. Please ensure that you have read and understood the current options disclosure document before entering into any options transactions.

SUMMARY

Volatility swaps are forward contracts on future realized stock volatility. Variance swaps are similar contracts on variance, the square of future volatility. Both of these instruments provide an easy way for investors to gain exposure to the future level of volatility.

Unlike a stock option, whose volatility exposure is contaminated by its stock-price dependence, these swaps provide pure exposure to volatility alone. You can use these instruments to speculate on future volatility levels, to trade the spread between realized and implied volatility, or to hedge the volatility exposure of other positions or businesses.

In this report we explain the properties and the theory of both variance and volatility swaps, first from an intuitive point of view and then more rigorously. The theory of variance swaps is more straightforward. We show how a variance swap can be theoretically replicated by a hedged portfolio of standard options with suitably chosen strikes, as long as stock prices evolve without jumps. The fair value of the variance swap is the cost of the replicating portfolio. We derive analytic formulas for theoretical fair value in the presence of realistic volatility skews. These formulas can be used to estimate swap values quickly as the skew changes.

We then examine the modifications to these theoretical results when reality intrudes, for example when some necessary strikes are unavailable, or when stock prices undergo jumps. Finally, we briefly return to volatility swaps, and show that they can be replicated by dynamically trading the more straightforward variance swap. As a result, the value of the volatility swap depends on the volatility of volatility itself.

Kresimir Demeterfi (212) 357-4611
Emanuel Derman (212) 902-0129
Michael Kamal (212) 357-3722
Joseph Zou (212) 902-9794

Acknowledgments: We thank Emmanuel Boussard, Llewellyn Connolly, Rustom Khandalavala, Cyrus Pirasteh, David Rogers, Emmanuel Roman, Peter Selman, Richard Sussman, Nicholas Warren and several of our clients for many discussions and insightful questions about volatility swaps.

Editorial: Barbara Dunn

Table of Contents

INTRODUCTION

A stock's volatility is the simplest measure of its riskiness or uncertainty. Formally, the volatility σ_R is the annualized standard deviation of the stock's returns during the period of interest, where the subscript R denotes the observed or "realized" volatility. This note is concerned with volatility swaps and other instruments suitable for trading volatility[1].

Why trade volatility? Just as stock investors think they know something about the direction of the stock market, or bond investors think they can foresee the probable direction of interest rates, so you may think you have insight into the level of future volatility. If you think current volatility is low, for the right price you might want to take a position that profits if volatility increases.

Investors who want to obtain pure exposure to the direction of a stock price can buy or sell short the stock. What do you do if you simply want exposure to a stock's volatility?

Stock options are impure: they provide exposure to both the direction of the stock price and its volatility. If you hedge the options according to Black-Scholes prescription, you can remove the exposure to the stock price. But delta-hedging is at best inaccurate because the real world violates many of the Black-Scholes assumptions: volatility cannot be accurately estimated, stocks cannot be traded continuously, transactions costs cannot be ignored, markets sometimes move discontinuously and liquidity is often a problem. Nevertheless, imperfect as they are, until recently options were the only volatility vehicle available.

Volatility Swaps

The easy way to trade volatility is to use *volatility swaps*, sometimes called *realized volatility forward contracts*, because they provide pure exposure to volatility (and only to volatility).

A stock volatility swap is a forward contract on annualized volatility. Its payoff at expiration is equal to

$$(\sigma_R - K_{vol}) \times N \qquad \textbf{(EQ 1)}$$

where σ_R is the realized stock volatility (quoted in annual terms) over the life of the contract, K_{vol} is the annualized volatility delivery price, and N is the notional amount of the swap in dollars per annualized volatility point. The holder of a volatility swap at expiration receives N dollars for every point by which the stock's realized volatility σ_R has

1. For a discussion of volatility as an asset class, see Derman, Kamal, Kani, McClure, Pirasteh, and Zou (1996).

exceeded the volatility delivery price K_{vol}. He or she is swapping a fixed volatility K_{vol} for the actual ("floating") future volatility $\sigma_{R.}$

The delivery price K_{vol} is typically quoted as a volatility, for example 30%. The notional amount is typically quoted in dollars per volatility point, for example, N = $250,000/(volatility point). As with all forward contracts or swaps, the fair value of volatility at any time is the delivery price that makes the swap currently have zero value.

The procedure for calculating the realized volatility should be clearly specified with respect to the following aspects:

- The source and observation frequency of stock or index prices – for example, using daily closing prices of the S&P 500 index;
- The annualization factor in moving from daily or hourly observations to annualized volatilities – for example, using 260 business days per year as a multiplicative factor in computing annualized variances from daily returns; and
- Whether the standard deviation of returns is calculated by subtracting the sample mean from each return, or by assuming a zero mean. The zero mean method is theoretically preferable, because it corresponds most closely to the contract that can be replicated by options portfolios. For frequently observed prices, the difference is usually negligible.

Who Can Use Volatility Swaps?

Volatility has several characteristics that make trading attractive. It is likely to grow when uncertainty and risk increase. As with interest rates, volatilities appear to revert to the mean; high volatilities will eventually decrease, low ones will likely rise. Finally, volatility is often negatively correlated with stock or index level, and tends to stay high after large downward moves in the market. Given these tendencies, several uses for volatility swaps follow.

Directional Trading of Volatility Levels. Clients who want to speculate on the future levels of stock or index volatility can go long or short realized volatility with a swap. This provides a much more direct method than trading and hedging options. For example, if you foresee a rapid decline in political and financial turmoil after a forthcoming election, a short position in volatility might be appropriate.

Trading the Spread between Realized and Implied Volatility Levels. As we will show later, the fair delivery price K_{vol} of a volatility swap is a value close to the level of current implied volatilities for options with the same expiration as the swap. Therefore, by unwinding

the swap before expiration, you can trade the spread between realized and implied volatility.

Hedging Implicit Volatility Exposure. There are several businesses that are implicitly short volatility:

- Risk arbitrageurs or hedge funds often take positions which assume that the spread between stocks of companies planning mergers will narrow. If overall market volatility increases, the merger may become less likely and the spread may widen.
- Investors following active benchmarking strategies may require more frequent rebalancing and greater transactions expenses during volatile periods.
- Portfolio managers who are judged against a benchmark have tracking error that may increase in periods of higher volatility.
- Equity funds are probably short volatility because of the negative correlation between index level and volatility. As global equity correlations have increased, diversification across countries has become a less effective portfolio hedge. Since volatility is one of the few parameters that tends to increase during global equity declines, a long volatility hedge may be appropriate, especially for financial businesses.

Variance Swaps

Although options market participants talk of volatility, it is *variance*, or volatility squared, that has more fundamental theoretical significance. This is so because the correct way to value a swap is to value the portfolio that replicates it, and the swap that can be replicated most reliably (by portfolios of options of varying strikes, as we show later) is a *variance swap*.

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$(\sigma_R^2 - K_{var}) \times N \qquad \textbf{(EQ 2)}$$

where σ_R^2 is the realized stock variance (quoted in annual terms) over the life of the contract, K_{var} is the delivery price for variance, and N is the notional amount of the swap in dollars per annualized volatility point squared. The holder of a variance swap at expiration receives N dollars for every point by which the stock's realized variance σ_R^2 has exceeded the variance delivery price K_{var}.

Though theoretically simpler, variance swaps are less commonly traded, and so their quoting conventions vary. The delivery price K_{var} can be quoted as a volatility squared, for example $(30\%)^2$. Similarly, for example, the notional amount can be expressed as \$100,000/(one volatility point)^2. The fair value of variance is the delivery price that makes the swap have zero value.

Outline

Most of this note will focus on the theory and properties of variance swaps, which provide similar volatility exposure to straight volatility swaps. Because of its fundamental role, variance can serve as the basic building block for constructing other volatility-dependent instruments. At the end, we will return to a discussion of the additional risks involved in replicating and valuing volatility swaps.

Section I presents an intuitive, Black-Scholes-based account of the fundamental strategy by which a variance swap can be replicated and valued. First, we show that the hedging of a (slightly) exotic stock option, the *log contract*, provides a payoff equal to the variance of the stock's returns under a fairly wide set of circumstances. Then, we explain how this exotic option itself can be replicated by a portfolio of standard stock options with a range of strikes, so that their market prices determine the cost of the variance swap. We also provide insight into the swap by showing, from a variety of viewpoints, how the apparently complex hedged log contract produces an instrument with the simple constant exposure to the realized variance of a variance swap.

Section II derives the same results much more rigorously and generally, without depending on the full validity of the Black-Scholes model. Though more difficult, this presentation is capable of much greater generalization.

In Section III, we provide a detailed numerical example of the valuation of a variance swap. Some practical issues concerning the choice of strikes are also discussed.

The fair value of the variance swap is determined by the cost of the replicating portfolio of options. This cost, especially for index options, is significantly affected by the volatility smile or skew. Therefore, we devote Section IV to the effects of the skew. In particular, for a skew linear in strike or linear in delta, we derive theoretical formulas that allow us to simply determine the approximate effect of the skew on the fair value of index variance swaps, without detailed numerical computation. The formulas and the intuition they provide are beneficial in rapidly estimating the effect of changes in the skew on swap values.

The fair value of a variance swap is based on (1) the ability to replicate a log contract by means of a portfolio of options with a (continuous) range of strikes, and (2) on classical options valuation theory, which assumes continuous stock price evolution. In practice, not all strikes are available, and stock prices can jump. Section V discusses the effects of these real limitations on pricing.

Finally, Section VI explains the risks involved in replicating a volatility contract. Since variance can be replicated relatively simply, it is useful to regard volatility as the square root of variance. From this point of view, volatility is itself a square-root derivative contract on variance. Thus, a volatility swap can be dynamically hedged by trading the underlying variance swap, and its value depends on the volatility of the underlying variance – that is, on the volatility of volatility.

Four appendices cover some more advanced mathematics. In Appendix A, we derive the details of the replication of a log contract by means of a continuum of option strikes. It also shows how the replication can be approximated in practice when only a discrete set of strikes are available. In Appendix B, we derive the approximate formulas for the value of an index variance contract in the presence of a volatility skew that varies linearly with strike. In Appendix C, we derive the analogous formulas for a skew varying linearly with the delta exposure of the options. Appendix D provides additional insight into the static and dynamic hedging of a volatility swap using the variance as an underlyer.

REPLICATING VARIANCE SWAPS: FIRST STEPS

In this section, we explain the replicating strategy that captures realized variance. The cost of implementing that strategy is the fair value of future realized variance.

The Intuitive Approach: Creating a Portfolio of Options Whose Variance Sensitivity is Independent of Stock Price

We approach variance replication by building on the reader's assumed familiarity with the standard Black-Scholes model. In the next section, we shall provide a more general proof that you can replicate variance, even when some of the Black-Scholes assumptions fail, as long as the stock price evolves continuously – that is, without jumps.

We ease the development of intuition by assuming here that the riskless interest rate is zero. Suppose at time t you own a standard call option of strike K and expiration T, whose value is given by the Black-Scholes formula $C_{BS}(S, K, \sigma\sqrt{\tau})$, where S is the current stock price, τ is the time to expiration $(T - t)$, σ is the return volatility of the stock, σ^2 is the stock's variance, and $v = \sigma^2\tau$ is the total variance of the stock to expiration. (We have written the option value as a function of $\sigma\sqrt{\tau}$ in order to make clear that all its dependence on both volatility and time to expiration is expressed in the combined variable $\sigma\sqrt{\tau}$.)

We will call the exposure of an option to a stock's variance V; it measures the change in value of the position resulting from a change in variance[2]. Figure 1a shows a graph of how V varies with stock price S, for each of three different options with strikes 80, 100 and 120. For each strike, the variance exposure V is largest when the option is at the money, and falls off rapidly as the stock price moves in or out of the money. V is closely related to the time sensitivity or time decay of the option, because, in the Black-Scholes formula with zero interest rate, options values depend on the total variance $\sigma^2\tau$.

If you want a long position in future realized variance, a single option is an imperfect vehicle: as soon as the stock price moves, your sensitivity to further changes in variance is altered. What you want is a portfo-

2. Here, we define the sensitivity $V = \frac{\partial C_{BS}}{\partial \sigma^2} = \frac{S\sqrt{\tau}}{2\sigma}\frac{\exp(-d_1^2/2)}{\sqrt{2\pi}}$, where $d_1 = \frac{\log(S/K) + (\sigma^2\tau)/2}{\sigma\sqrt{\tau}}$. We will sometimes refer to V as "variance vega". Note that d_1 depends only on the two combinations S/K and $\sigma\sqrt{\tau}$. V decreases extremely rapidly as S leaves the vicinity of the strike K.

lio whose sensitivity to realized variance is independent of the stock price S.

To obtain a portfolio that responds to volatility or variance independent of moves in the stock price, you need to combine options of many strikes. What combination of strikes will give you such undiluted variance exposure?

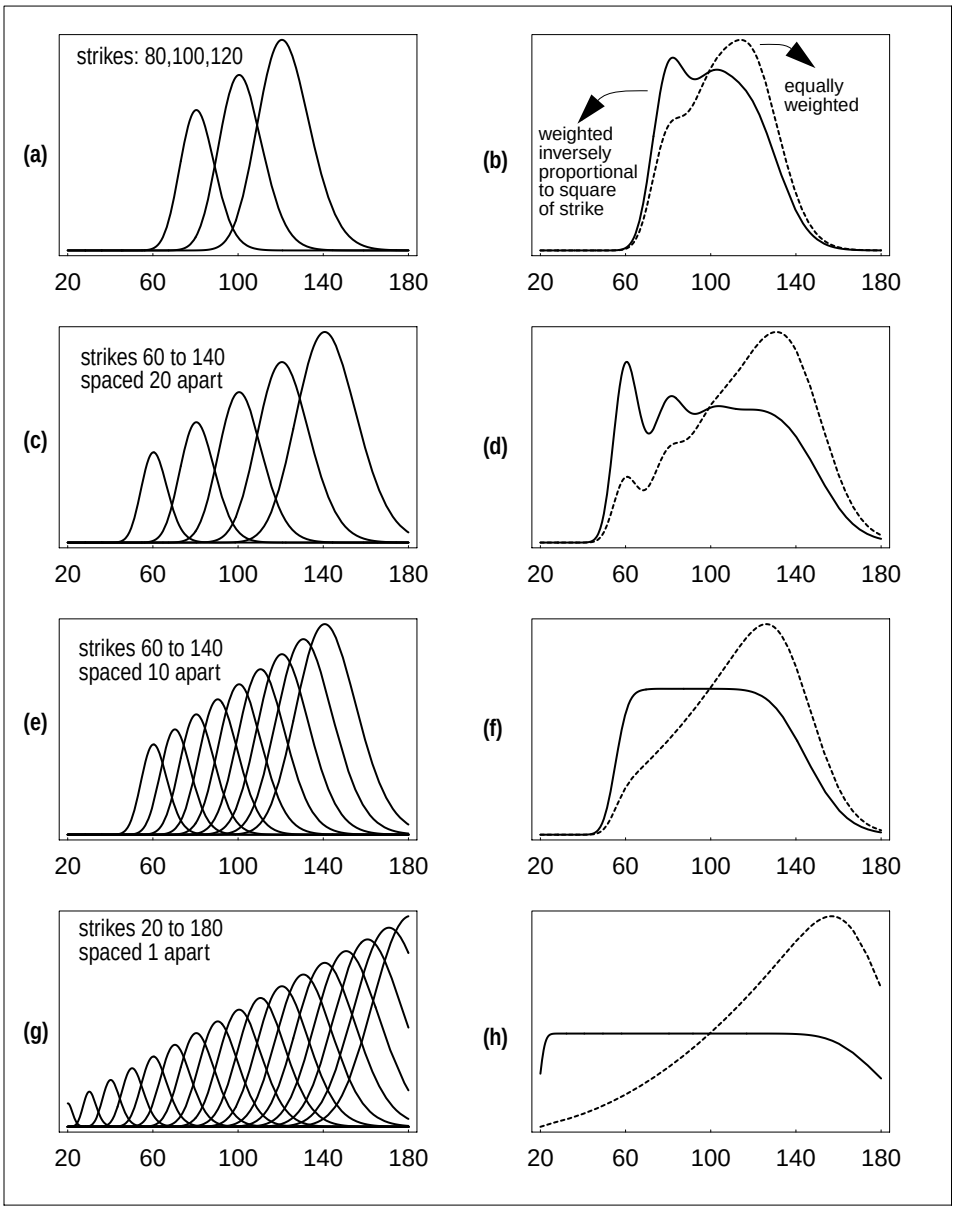
Figure 1b shows the variance exposure for the portfolio consisting of all three option strikes in Figure 1a. The dotted line represents the sum of equally weighted strikes; the solid line represents the sum with weights in inverse proportion to the square of their strike. Figures 1c, e and g show the individual sensitivities to variance of increasing numbers of options, each panel having the options more closely spaced. Figures 1d, f and h show the sensitivity for the equally-weighted and strike-weighted portfolios. Clearly, the portfolio with weights inversely proportional to K^2 produces a V that is virtually independent of stock price S, as long as S lies inside the range of available strikes and far from the edge of the range, and provided the strikes are distributed evenly and closely.

Appendix A provides a mathematical derivation of the requirement that options be weighted inversely proportional to K^2 in order to achieve constant V. You can also understand this intuitively. As the stock price moves up to higher values, each additional option of higher strike in the portfolio will provide an additional contribution to V proportional to that strike. This follows from the formula in footnote 2, and you can observe it in the increasing height of the V-peaks in Figure 1a. An option with higher strike will therefore produce a V contribution that increases with S. In addition, the contributions of all options overlap at any definite S. Therefore, to offset this accumulation of S-dependence, one needs diminishing amounts of higher-strike options, with weights inversely proportional to K^2.

If you own a portfolio of options of all strikes, weighted in inverse proportion to the square of the strike level, you will obtain an exposure to variance that is independent of stock price, just what is needed to trade variance. What does this portfolio of options look like, and how does trading it capture variance?

Consider the portfolio $\Pi(S, \sigma\sqrt{\tau})$ of options of all strikes K and a single expiration τ, weighted inversely proportional to K^2. Because out-of-

FIGURE 1. The variance exposure, V, of portfolios of call options of different strikes as a function of stock price S. Each figure on the left shows the individual V_i contributions for each option of strike K_i. The corresponding figure on the right shows the sum of the contributions, weighted two different ways; the dotted line corresponds to an equally-weighted sum of options; the solid line corresponds to weights inversely proportional to K^2, and becomes totally independent of stock price S inside the strike range



the-money options are generally more liquid, we employ put options $P(S, K, \sigma\sqrt{\tau})$ for strikes K varying continuously from zero up to some reference price S_*, and call options $C(S, K, \sigma\sqrt{\tau})$ for strikes varying continuously from S_* to infinity[3]. You can think of S_* as the approximate at-the-money forward stock level that marks the boundary between liquid puts and liquid calls.

At expiration, when $t = T$, one can show that the sum of all the payoff values of the options in the portfolio is simply

$$\Pi(S_T, 0) = \frac{S_T - S_*}{S_*} - \log\left(\frac{S_T}{S_*}\right) \qquad \textbf{(EQ 3)}$$

where log( ) denotes the natural logarithm function, and S_T is the terminal stock price.

Similarly, at time t you can sum all the Black-Scholes options values to show that the total portfolio value is

$$\Pi(S, \sigma\sqrt{\tau}) = \frac{S - S_*}{S_*} - \log\left(\frac{S}{S_*}\right) + \frac{\sigma^2\tau}{2} \qquad \textbf{(EQ 4)}$$

where S is the stock price at time t. Note how little the value of the portfolio before expiration differs from its value at expiration at the same stock price. The only difference is the additional value due to half the total variance $\sigma^2\tau$.

Clearly, the variance exposure of Π is

$$V = \frac{\tau}{2} \qquad \textbf{(EQ 5)}$$

To obtain an initial exposure of \$1 per volatility point squared, you need to hold $(2/T)$ units of the portfolio Π. From now on, we will use Π to refer to the value of this new portfolio, namely

3. Formally, the expression for the portfolio is given by

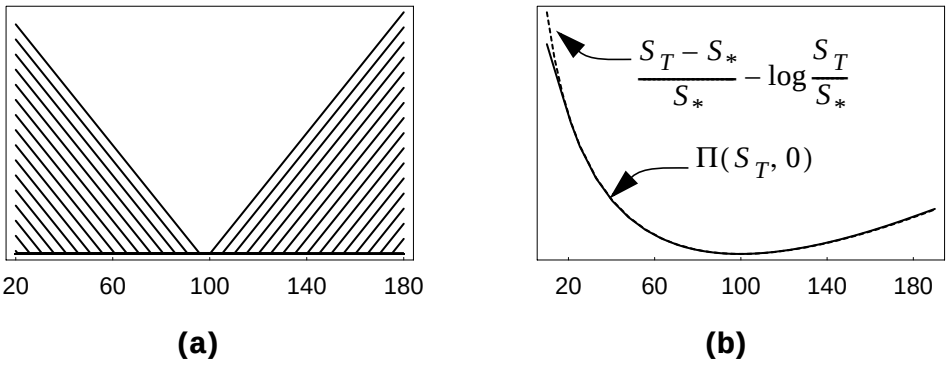
$$\Pi(S, \sigma\sqrt{\tau}) = \sum_{K > S_*} \frac{1}{K^2} C(S, K, \sigma\sqrt{\tau}) + \sum_{K < S_*} \frac{1}{K^2} P(S, K, \sigma\sqrt{\tau})$$

$$\Pi(S, \sigma\sqrt{\tau}) = \frac{2}{T}\left[\frac{S - S_*}{S_*} - \log\left(\frac{S}{S_*}\right)\right] + \frac{\tau}{T}\sigma^2 \qquad \textbf{(EQ 6)}$$

The first term in the payoff in Equation 6, $(S - S_*)/S_*$, describes $1/S_*$ forward contracts on the stock with delivery price S_*. It is not really an option; its value represents a long position in the stock (value S) and a short position in a bond (value S_*), which can be statically replicated, once and for all, without any knowledge of the stock's volatility. The second term, $-\log(S/S_*)$, describes a short position in a *log contract*[4] with reference value S_*, a so-called exotic option whose payoff is proportional to the log of the stock at expiration, and whose correct hedging depends on the volatility of the stock. All of the volatility sensitivity of the weighted portfolio of options we have created is contained in the log contract.

Figure 2 graphically illustrates the equivalence between (1) the summed, weighted payoffs of puts and calls and (2) a long position in a forward contract and a short position in a log contract.

FIGURE 2. An example that illustrates the equivalence at expiration between (1) $\Pi(S_T,0)$ the weighted sum of puts and calls, with weight inversely proportional to the square of the strike, and (2) the payoff of a long position in a forward contract and a short position in a log contract. (a) Individual contributions to the payoff from put options with all integer strikes from 20 to 99, and call options with all integer strikes from 100 to 180. (b) The payoff of 1/100 of a long position in a forward contract with delivery price 100 and one short position in a log contract with reference value 100.



4. The log contract was first discussed in Neuberger (1994). See also Neuberger (1996).

Trading Realized Volatility with a Log Contract

For now, assume that we are in a Black-Scholes world where the implied volatility σ_I is the estimate of future realized volatility. If you take a position in the portfolio Π, the fair value you should pay at time $t = 0$ when the stock price is S_0 is

$$\Pi_0 = \frac{2}{T}\left[\frac{S_0 - S_*}{S_*} - \log\left(\frac{S_0}{S_*}\right)\right] + \sigma_I^2$$

At expiration, if the realized volatility turns out to have been σ_R, the initial fair value of the position captured by delta-hedging would have been

$$\Pi_0 = \frac{2}{T}\left[\frac{S_0 - S_*}{S_*} - \log\left(\frac{S_0}{S_*}\right)\right] + \sigma_R^2$$

The net payoff on the position, hedged to expiration, will be

$$\text{payoff} = (\sigma_R^2 - \sigma_I^2) \qquad \textbf{(EQ 7)}$$

Looking back at Equation 2, you will see that *by rehedging the position in log contracts, you have, in effect, been the owner of a position in a variance swap with fair strike* $K_{var} = \sigma_I^2$ *and face value \$1.* You will have profited (or lost) if realized volatility has exceeded (or been exceeded by) implied volatility.

The Vega, Gamma and Theta of a Log Contract

In Equation 6 we showed that, in a Black-Scholes world with zero interest rates and zero dividend yield, the portfolio of options whose variance vega is independent of the stock price S can be written

$$\Pi(S, \sigma, t, T) = \frac{2}{T}\left[\frac{S - S_*}{S_*} - \log\left(\frac{S}{S_*}\right)\right] + \frac{(T-t)}{T}\sigma^2$$

The $(S - S_*)$ term represents a long position in the stock and a short position in a bond, both of which can be statically hedged with no dependence on volatility. In contrast, the log( ) term needs continual dynamic rehedging. Therefore, let us concentrate on the log contract term alone, whose value at time t for a logarithmic payoff at time T is

$$L(S, \sigma, t, T) = -\frac{2}{T}\log\left(\frac{S}{S_*}\right) + \frac{(T-t)}{T}\sigma^2 \qquad \textbf{(EQ 8)}$$

The sensitivities of the value of this portfolio are precisely appropriate for trading variance, as we now show.

The variance vega of the portfolio in Equation 8 is

$$V = \left(\frac{T-t}{T}\right) \quad \textbf{(EQ 9)}$$

The exposure to variance is equal to 1 at $t = 0$, and decreases linearly to zero as the contract approaches expiration.

The time decay of the log contract, the rate at which its value changes if the stock price remains unchanged as time passes, is

$$\theta = -\frac{1}{T}\sigma^2 \quad \textbf{(EQ 10)}$$

The contract loses time value at a constant rate proportional to its variance, so that at expiration, all the initial variance has been lost.

The log contract's exposure to stock price is

$$\Delta = -\frac{2}{T}\frac{1}{S}$$

shares of stock. That is, since each share of stock is worth S, you need a constant long position in $\$(2/T)$ worth of stock to be hedged at any time.

The gamma of the portfolio, the rate at which the exposure changes as the stock price moves, is

$$\Gamma = \frac{2}{T}\frac{1}{S^2} \quad \textbf{(EQ 11)}$$

Gamma is a measure of the risk of hedging an option. The log contract's gamma, being the sum of the gammas of a portfolio of puts and calls, is a smoother function of S than the sharply peaked gamma of a single option.

Equations 10 and 11 can be combined to show that

$$\theta + \frac{1}{2}\Gamma S^2\sigma^2 = 0 \quad \textbf{(EQ 12)}$$

Equation 12 is the essence of the Black-Scholes options pricing theory. It states that the disadvantage of negative theta (the decrease in value with time to expiration) is offset by the benefit of positive gamma (the curvature of the payoff).

Imperfect Hedges

It takes an infinite number of strikes, appropriately weighted, to replicate a variance swap. In practice, this isn't possible, even when the stock and options market satisfy all the Black-Scholes assumptions: there are only a finite number of options available at any maturity. Figure 1 illustrates that a finite number of strikes fails to produce a uniform V as the stock price moves outside the range of the available strikes. As long as the stock price remains within the strike range, trading the imperfectly replicated log contract will allow variance to accrue at the correct rate. Whenever the stock price moves outside, the reduced vega of the imperfectly replicated log contract will make it less responsive than a true variance swap.

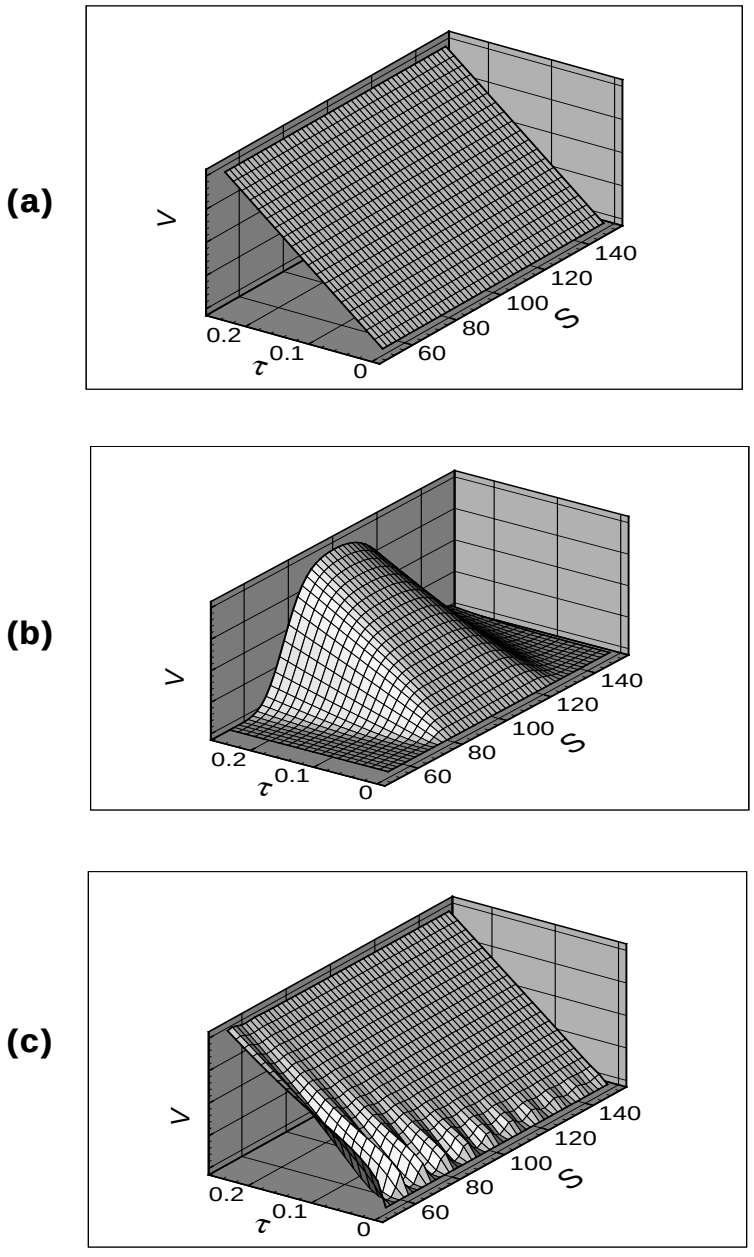
Figure 3 shows how the variance vega of a three-month variance swap is affected by imperfect replication. Figure 3a shows the ideal variance vega that results from a portfolio of puts and calls of all strikes from zero to infinity, weighted in inverse proportion to the strike squared. Here the variance vega is independent of stock price level, and decreases linearly with time to expiration, as expected for a swap whose value is proportional to the remaining variance $\sigma^2\tau$ at any time. Figure 3b shows strikes from $75 to $125, uniformly spaced $1 apart. Here, deviation from constant variance vega develops at the tail of the strike range, and the deviation is greater at earlier times. Finally, Figure 3c shows the vega for strikes from $20 to $200, spaced $10 apart. Now, although the range of strikes is greater, the coarser spacing causes the vega surface to develop corrugations between strike values that grow more pronounced closer to expiration.

The Limitations of the Intuitive Approach

A variance swap has a payoff proportional to realized variance. In this section, assuming the Black-Scholes world for stock and options markets, we have shown that the dynamic, continuous hedging of a log contract produces a payoff whose value is proportional to future realized variance. We have also shown that you can use a portfolio of appropriately weighted puts and calls to approximate a log contract.

The somewhat intuitive derivations in this section have assumed that interest rates and dividend yields are zero, but it is not hard to generalize them. We have also assumed that all the Black-Scholes assumptions hold. In practice, in the presence of an implied volatility skew, it is difficult to extend these argument clearly. Therefore, we move on to a more general and rigorous derivation of the value of variance swaps based on replication. Many of the results will be similar, but the conditions under which they hold, and the correct answers when they do not hold, will be more easily understandable.

FIGURE 3. The variance vega, V, of a portfolio of puts and calls, weighted inversely proportional to the square of the strike level, and chosen to replicate a three-month variance swap. (a) An infinite number of strikes. (b) Strikes from $75 to $125, uniformly spaced $1 apart. (c) Strikes from $20 to $200, uniformly spaced $10 apart.



REPLICATING VARIANCE SWAPS: GENERAL RESULTS

In the previous section, we explained how to replicate a variance swap by means of a portfolio of options whose payoffs approximate a log contract. Although our explanation depended on the validity of the Black-Scholes model, the result – that the dynamic hedging of a log contract captures realized volatility – holds true more generally. The only assumption we will make about the future underlyer evolution is that it is diffusive, or continuous – this means that no jumps are allowed. (In a later section, we will consider the effects of discontinuous stock price movements on the success of the replication.) Therefore, we assume that the stock price evolution is given by

$$\frac{dS_t}{S_t} = \mu(t, ...)dt + \sigma(t, ...)dZ_t \qquad \textbf{(EQ 13)}$$

Here, we assume that the drift μ and the continuously-sampled volatility σ are arbitrary functions of time and other parameters. These assumptions include, but are not restricted to, implied tree models in which the volatility is a function $\sigma(S, t)$ of stock price and time only. For simplicity of presentation, we assume the stock pays no dividends; allowing for dividends does not significantly alter the derivation.

The theoretical definition of realized variance for a given price history is the continuous integral

$$V = \frac{1}{T}\int_0^T \sigma^2(t, \ldots)dt \qquad \textbf{(EQ 14)}$$

This is a good approximation to the variance of daily returns used in the contract terms of most variance swaps.

Conceptually, valuing a variance forward contract or "swap" is no different than valuing any other derivative security. The value of a forward contract F on future realized variance with strike K is the expected present value of the future payoff in the risk-neutral world:

$$F = E[e^{-rT}(V - K)] \qquad \textbf{(EQ 15)}$$

Here r is the risk-free discount rate corresponding to the expiration date T, and $E[\,]$ denotes the expectation.

The fair delivery value of future realized variance is the strike K_{var} for which the contract has zero present value:

$$K_{var} = E[V] \qquad \textbf{(EQ 16)}$$

If the future volatility in Equation 13 is specified, then one approach for calculating the fair price of variance is to directly calculate the risk-neutral expectation

$$K_{var} = \frac{1}{T} E\left[\int_0^T \sigma^2(t, \ldots) dt\right] \quad \textbf{(EQ 17)}$$

No one knows with certainty the value of future volatility. In implied tree models[5], the so-called local volatility $\sigma(S, t)$ consistent with all current options prices is extracted from the market prices of traded stock options. You can then use simulation to calculate the fair variance K_{var} as the average of the experienced variance along each simulated path consistent with the risk-neutral stock price evolution given of Equation 13, where the drift μ is set equal to the riskless rate.

The above approach is good for valuing the contract, but it does not provide insight into how to replicate it. The essence of the replication strategy is to devise a position that, over the next instant of time, generates a payoff proportional to the incremental variance of the stock during that time[6].

By applying Ito's lemma to $\log S_t$, we find

$$d(\log S_t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dZ_t \quad \textbf{(EQ 18)}$$

Subtracting Equation 18 from Equation 13, we obtain

$$\frac{dS_t}{S_t} - d(\log S_t) = \frac{1}{2}\sigma^2 dt \quad \textbf{(EQ 19)}$$

in which all dependence on the drift μ has cancelled. Summing Equation 19 over all times from 0 to *T*, we obtain the continuously-sampled variance

5. See, for example, Derman and Kani (1994), Dupire (1994) and Derman, Kani and Zou (1996).
6. This approach was first outlined in Derman, Kamal, Kani, and Zou (1996). For an alternative discussion, see Carr and Madan (1998).

$$V \equiv \frac{1}{T}\int_0^T \sigma^2 dt$$
$$= \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}\right] \quad \text{(EQ 20)}$$

This mathematical identity dictates the replication strategy for variance. The first term in the brackets can be thought of as the net outcome of continuous rebalancing a stock position so that it is always instantaneously long $1/S_t$ shares of stock worth \$1. The second term represents a *static* short position in a contract which, at expiration, pays the logarithm of the total return. Following this continuous rebalancing strategy captures the realized variance of the stock from inception to expiration at time *T*. Note that no expectations or averages have been taken – Equation 20 guarantees that variance can be captured no matter which path the stock price takes, as long as it moves continuously.

Valuing and Pricing the Variance Swap

Equation 20 provides another method for calculating the fair variance. Instead of averaging over future variances, as in Equation 17, one can take the expected risk-neutral value of the right-hand side of Equation 20 to obtain the cost of replication directly, so that

$$K_{var} = \frac{2}{T}E\left[\int_0^T \frac{dS_t}{S_t} - \log\frac{S_T}{S_0}\right] \quad \text{(EQ 21)}$$

The expected value of the first term in Equation 21 accounts for the cost of rebalancing. In a risk-neutral world with a constant risk-free rate *r*, the underlyer evolves according to:

$$\frac{dS_t}{S_t} = r\,dt + \sigma(t, \ldots)dZ \quad \text{(EQ 22)}$$

so that the risk-neutral price of the rebalancing component of the hedging strategy is given by

$$E\left[\int_0^T \frac{dS_t}{S_t}\right] = rT \quad \text{(EQ 23)}$$

This equation represents the fact that a shares position, continuously rebalanced to be worth \$1, has a forward price that grows at the riskless rate.

As there are no actively traded log contracts for the second term in Equation 21, one must duplicate the log payoff, at all stock price levels at expiration, by decomposing its shape into linear and curved components, and then duplicating each of these separately. The linear component can be duplicated with a forward contract on the stock with delivery time T; the remaining curved component, representing the quadratic and higher order contributions, can be duplicated using standard options with all possible strike levels and the same expiration time T.

For practical reasons we want to duplicate the log payoff with liquid options – that is, with a combination of out-of-the-money calls for high stock values and out-of-the-money puts for low stock values. We introduce a new arbitrary parameter S_* to define the boundary between calls and puts. The log payoff can then be rewritten as

$$\log\frac{S_T}{S_0} = \log\frac{S_T}{S_*} + \log\frac{S_*}{S_0} \qquad \textbf{(EQ 24)}$$

The second term $\log(S_*/S_0)$ is constant, independent of the final stock price S_T, so only the first term has to be replicated.

The following mathematical identity, which holds for all future values of S_T, suggests the decomposition of the log-payoff:

$$\begin{aligned} -\log\frac{S_T}{S_*} &= -\frac{S_T - S_*}{S_*} && \text{(forward contract)} \\ &+ \int_0^{S_*} \frac{1}{K^2}\,\mathrm{Max}(K - S_T, 0)\,dK && \text{(put options)} \\ &+ \int_{S_*}^{\infty} \frac{1}{K^2}\,\mathrm{Max}(S_T - K, 0)\,dK && \text{(call options)} \end{aligned} \qquad \textbf{(EQ 25)}$$

Equation 25 represents the decomposition of a log payoff into a portfolio consisting of:

- a short position in $(1/S_*)$ forward contracts struck at S_*;
- a long position in $(1/K^2)$ put options struck at K, for all strikes from 0 to S_*; and
- a similar long position in $(1/K^2)$ call options struck at K, for all strikes from S_* to ∞.

All contracts expire at time T. Figure 4 shows this decomposition schematically.

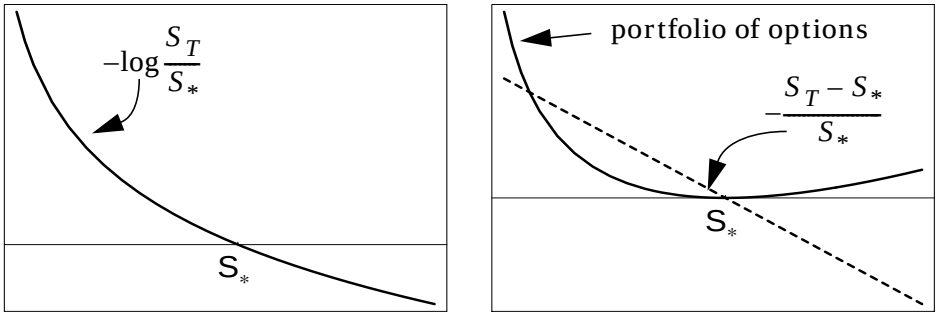
The fair value of future variance can be related to the initial fair value of each term on the right hand side of Equation 21. By using the identities in Equations 23 and 25, we obtain

$$K_{var} = \frac{2}{T}\left(rT - \left(\frac{S_0}{S_*} e^{rT} - 1 \right) - \log \frac{S_*}{S_0} + e^{rT} \int_0^{S_*} \frac{1}{K^2} P(K)\, dK + e^{rT} \int_{S_*}^{\infty} \frac{1}{K^2} C(K)\, dK \right) \qquad \text{(EQ 26)}$$

where $P(K)$ and $(C(K))$, respectively, denote the current fair value of a put and call option of strike K. If you use the market prices of these options, you obtain an estimate of the current market price of future variance.

This approach to the fair value of future variance is the most rigorous from a theoretical point of view, and makes less assumptions than our intuitive treatment in the section on page 6. Equation 26 makes precise the intuitive notion that implied volatilities can be regarded as the market's expectation of future realized volatilities. It provides a direct connection between the market cost of options and the strategy for capturing future realized volatility, *even when there is an implied volatility skew* and the simple Black-Scholes formula is invalid.

FIGURE 4. Replication of the log payoff. (a) The payoff of a short position in a log contract at expiration. (b) Dashed line: the linear payoff at expiration of a forward contract with delivery price S_*; Solid line: the curved payoff of calls struck above S_* and puts struck below S_*. Each option is weighted by the inverse square of its strike. The sum of the payoffs for the dashed and solid lines provide the same payoff as the log contract.



(a) **(b)**

AN EXAMPLE OF A VARIANCE SWAP

We now present a detailed practical example. Suppose you want to price a swap on the realized variance of the daily returns of some hypothetical equity index. The fair delivery variance is determined by the cost of the replicating strategy discussed in the previous section. If you could buy options of all strikes between zero and infinity, the fair variance would be given by Equation 26 with some choice of S_*, say $S_* = S_0$. In practice, however, only a small set of discrete option strikes is available, and using Equation 26 with only a few strikes leads to appreciable errors. Here we suggest a better approximation.

We start with the definition of fair variance given by Equation 21, which can be written as

$$K_{var} \equiv \frac{2}{T} E\left[\int_0^T \frac{dS_t}{S_t} - \frac{S_T - S_*}{S_*} - \log\frac{S_*}{S_0} + \frac{S_T - S_*}{S_*} - \log\frac{S_T}{S_*}\right]$$

Taking expectations, this becomes

$$K_{var} = \frac{2}{T}\left[rT - \left(\frac{S_0}{S_*}e^{rT} - 1\right) - \log\frac{S_*}{S_0}\right] + e^{rT}\Pi_{CP} \qquad \textbf{(EQ 27)}$$

where Π_{CP} is the present value of the portfolio of options with payoff at expiration given by

$$f(S_T) = \frac{2}{T}\left(\frac{S_T - S_*}{S_*} - \log\frac{S_T}{S_*}\right) \qquad \textbf{(EQ 28)}$$

Suppose that you can trade call options with strikes K_{ic} such that $K_0 = S_* < K_{1c} < K_{2c} < K_{3c} < \ldots$ and put options with strikes K_{ip} such that $K_0 = S_* > K_{1p} > K_{2p} > K_{3p} > \ldots$

In Appendix A we derive the formula that determines how many options of each strike you need in order to approximate the payoff $f(S_T)$ by piece-wise linear options payoffs. The procedure in Appendix A guarantees that these payoffs will always exceed or match the value of the log contract, but never be worth less. Once these weights are calculated, Π_{CP} is obtained from

$$\Pi_{CP} = \sum_i w(K_{ip})P(S, K_{ip}) + \sum_i w(K_{ic})C(S, K_{ic}) \qquad \textbf{(EQ 29)}$$

We now illustrate this procedure with a concrete numerical example.

TABLE 1. The portfolio of European-style put and call options used for calculating the cost of capturing realized variance in the presence of the implied volatility skew with a discrete set of options strikes.

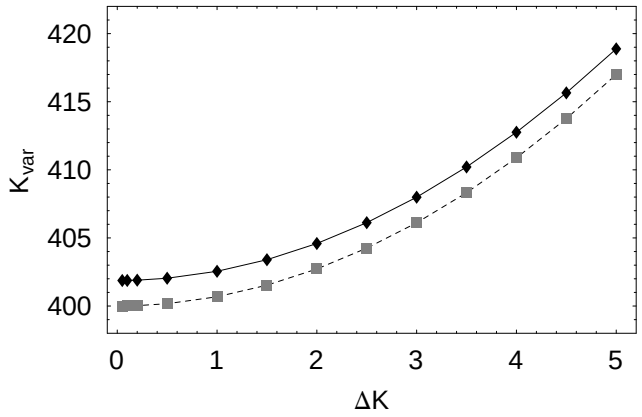
	Strike	Volatility	Weight	Value per Option	Contribution
PUTS	50	30	163.04	0.000002	0.0004
	55	29	134.63	0.00003	0.0035
	60	28	113.05	0.0002	0.0241
	65	27	96.27	0.0013	0.1289
	70	26	82.98	0.0067	0.5560
	75	25	72.26	0.0276	1.9939
	80	24	63.49	0.0958	6.0829
	85	23	56.23	0.2854	16.0459
	90	22	50.15	0.7384	37.0260
	95	21	45.00	1.6747	75.3616
	100	20	20.98	3.3537	70.3615
CALLS	100	20	19.63	4.5790	89.8691
	105	19	36.83	2.2581	83.1580
	110	18	33.55	0.8874	29.7752
	115	17	30.69	0.2578	7.9130
	120	16	28.19	0.0501	1.4119
	125	15	25.98	0.0057	0.1476
	130	14	24.02	0.0003	0.0075
	135	13	22.27	0.000006	0.0001
				TOTAL COST	**419.8671**

Assume that the index level S_0 is 100, the continuously compounded annual riskless interest rate r is 5%, the dividend yield is zero, and the maturity of the variance swap is three months (T = 0.25. Suppose that

you can buy options with strikes in the range from 50 to 150, uniformly spaced 5 points apart. We assume that at-the-money implied volatility is 20%, with a skew such that the implied volatility increases by 1 volatility point for every 5 point decrease in the strike level. In Table 1 we provide the list of strikes and their corresponding implied volatilities. We then show the weights, the value of each individual option and the contribution of each strike level to the total cost of the portfolio. At the bottom of the table we show the total cost of the options portfolio, $\Pi_{CP} = 419.8671$. It is clear from Table 1 that most of the cost comes from options with strikes near the spot value. Although the number of options which are far out of the money is large, their value is small and contributes little to the total cost.

The cost of capturing variance is now simply calculated using Equation 27 with the result $K_{var} = (20.467)^2$. This is not strictly the fair variance; because the procedure of approximating the log contract in Appendix A always over-estimates the value of the log contract, this value is higher than the true theoretical value for the fair variance obtained by approximating the log contract with a continuum of strikes. In Figure 5 we illustrate the cost of variance as function of the spacing between strikes, for two cases, with and without a volatility skew. You can see that as the spacing between strikes approaches zero, the cost of capturing variance approaches the theoretically fair variance.

FIGURE 5. Convergence of K_{var}, the cost of capturing variance with a discrete set of strikes, towards the fair value of variance as a function of ΔK, the spacing between strikes. The line with square symbols shows the convergence for no skew, with all implied volatilities at the same value of 20%. The theoretical fair variance for $\Delta K = 0$ is then $(20)^2 = 400$. The line with diamond symbols shows similar convergence to a higher fair variance of about 402, the extra contribution coming from the effect of the skew.



EFFECTS OF THE VOLATILITY SKEW

The general strategy discussed in the previous section can be used to determine the fair variance and the hedging portfolio from the set of available options and their implied volatilities. Here we discuss the effects of a volatility skew on the fair variance. We assume that there is no term structure and consider two different skew parameterizations, both of which resemble typical index skews. The first is a skew that varies linearly with the strike of the option, the second a skew that varies linearly with the Black-Scholes delta. In both cases we will compare the numerically correct value of fair variance, computed from Equation 26, with an approximate analytic formula that we derive. This formula provides a good rule of thumb for a quick estimate of the impact of the volatility skew on the fair variance.

Skew Linear in Strike

We first consider a skew for which the implied volatility varies linearly with strike, so that

$$\Sigma(K) = \Sigma_0 - b\frac{K - S_F}{S_F} \qquad \textbf{(EQ 30)}$$

Here Σ_0 is the implied volatility of an option struck at the forward. The steepness of the skew is determined by the slope b, with a positive value indicating a higher volatility for strikes below the forward. Note that this parametrization cannot hold for all strikes, because, for a large enough value of K, the implied volatility would become negative[7]. A value of $b = 0.2$ means that the implied volatility corresponding to a strike 10% below the forward, for example, is 2 volatility points higher than Σ_0. In Appendix B we derive the following approximate formula for the fair variance of the contract with time to expiration T:

$$K_{var} \approx \Sigma_0^2(1 + 3Tb^2 + ...) \qquad \textbf{(EQ 31)}$$

The skew increases the value of the fair variance above the at-the-money-forward level of volatility, and the size of the increase is proportional to time to maturity and the square of the skew slope. (Note that b in Equation 30 has the same dimension as volatility, so that b^2T is a dimensionless parameter, and therefore a natural candidate for the order of magnitude of the percentage correction to K_{var}. Note also that there is no term $b\sqrt{T}$ in Equation 31. This approximation works best for short maturities and skews that are not too steep.

7. Note that for large values of K, where this parameterization is invalid, the options prices in Equation 26 are negligible and therefore do not affect the value of the fair variance.

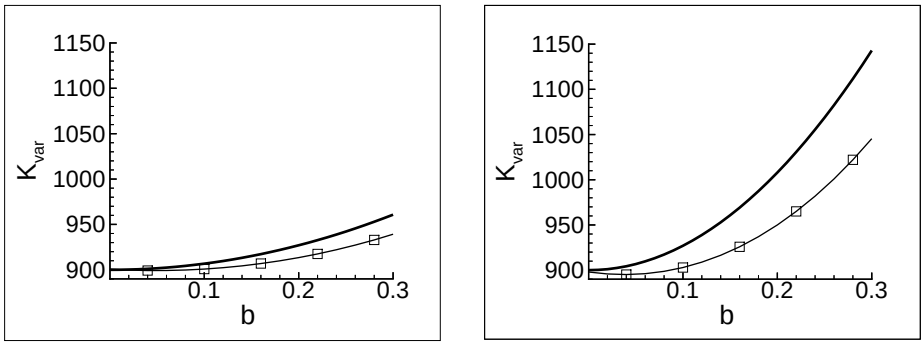
In Table 2 we compare the exact results for fair variance, computed numerically, with the approximate values given by the analytic formula in Equation 31.

TABLE 2. Comparison of the exact fair variance, computed numerically, with the approximate analytic formula of Equation 31. We assume Σ_0 = 30%, S = 100, the continuously compounded annual discount rate r = 5%, zero dividend yield, and use strikes evenly spaced one point apart from K = 10 to K = 200 to replicate the log payoff.

Skew Slope b	T = 3 months		T = 1 year	
	Exact Value	Analytic Approximation	Exact Value	Analytic Approximation
0.0	$(30.01)^2$	$(30.00)^2$	$(29.97)^2$	$(30.00)^2$
0.1	$(30.01)^2$	$(30.11)^2$	$(30.05)^2$	$(30.45)^2$
0.2	$(30.22)^2$	$(30.44)^2$	$(30.82)^2$	$(31.75)^2$
0.3	$(30.65)^2$	$(30.99)^2$	$(32.33)^2$	$(33.81)^2$

Figure 6 contains a graph of these results. We see excellent agreement in the case of the three-month variance swap, and reasonable agreement for one year.

FIGURE 6. Comparison of the exact value of fair variance, K_{var}, with the approximate value from the formula of Equation 31, as a function of the skew slope b. The thin line with squares shows the exact values obtained by replicating the log-payoff. The thick line depicts the approximate value given by Equation 31. (a) three-month variance swap. (b) one-year variance swap.



(a) **(b)**

Skew Linear in Delta

Next we consider a skew that varies linearly with the Black-Scholes delta of the option, so that:

$$\Sigma(\Delta_p) = \Sigma_0 + b\left(\Delta_p + \frac{1}{2}\right) \qquad \textbf{(EQ 32)}$$

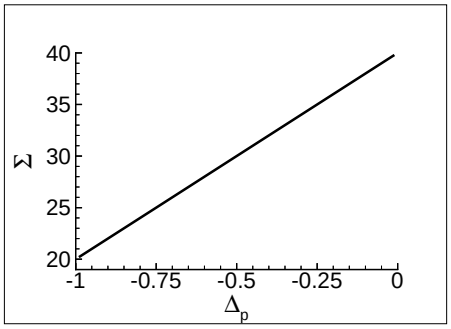
Here Δ_p is the Black-Scholes exposure of a put option, given by $\Delta_p = -N(-d_1)$, where d_1 is defined in Footnote 2, Σ_0 is the implied volatility of a "50-delta" put option and b is the slope of the skew – that is, the change in the skew per unit delta. This parameter b is not the same as the b in the previous section. In particular, there is an implicit dependence on the time to expiration in the formula of Equation 32, because of the Δ_p term, which was absent from Equation 30. Since Δ_p is bounded, the implied volatility is always positive provided $b < 2\Sigma_0$. This restriction is irrelevant, since Equation 32 leads to arbitrage violation before b reaches this limit. In practice, this parameterization leads to more realistic skews than those produced by the linear-strike formula.

Appendix C presents a detailed derivation of the following approximate formula for the fair variance of the contract with time to expiration T:

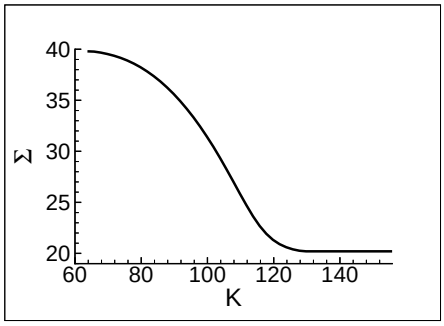
$$K_{var} \approx \Sigma_0^2\left(1 + \frac{1}{\sqrt{\pi}} b\sqrt{T} + \frac{1}{12}\frac{b^2}{\Sigma_0^2} + \ldots.\right) \qquad \textbf{(EQ 33)}$$

Here, in contrast to the skew linear in strike, the first-order correction is of magnitude $b\sqrt{T}$, because a variation linear in delta about the at-the-money-forward strike is not equivalent to a variation linear in strike.

FIGURE 7. (a) A volatility skew that varies linearly in delta. (b) The corresponding skew plotted as a function of strike. We have assumed that the stock price S is 100, the continuously compounded annual discount rate r is 5%, the term to maturity is three months, and the skew slope is 0.2.



(a)



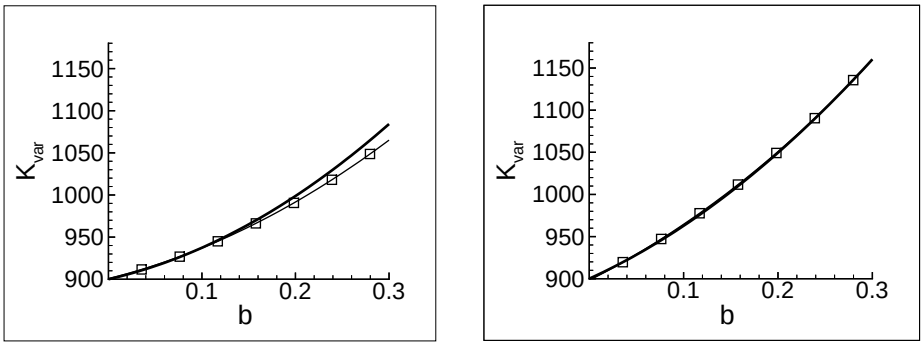
(b)

First, we convert the skew by delta in Equation 32 into a skew by strike, as displayed in Figure 7. Again, we compare the exact results computed according to Appendix A with the approximate values given by Equation 33. In Table 3 we compare the results for fair variance, computed numerically, with the approximate values given by the analytic formula in Equation 33. The analytic formula works very well for the three-month variance swap, and truly impressively for the one-year swap, as displayed in Figure 8.

TABLE 3. Comparison of the fair variance, computed numerically, with the approximate analytic formula of Equation 33. We assume Σ_0 = 30%, S = 100, the continuously compounded annual discount rate r = 5%, zero dividend yield, and use strikes evenly spaced one point apart from K = 10 to K = 200 to replicate the log payoff.

Skew Slope b	T = 3 months		T = 1 year	
	Exact Value	Analytic Approximation	Exact Value	Analytic Approximation
0.0	$(30.01)^2$	$(30.00)^2$	$(29.97)^2$	$(30.00)^2$
0.1	$(30.61)^2$	$(30.62)^2$	$(31.06)^2$	$(31.03)^2$
0.2	$(31.49)^2$	$(31.60)^2$	$(32.42)^2$	$(32.40)^2$
0.3	$(32.64)^2$	$(32.93)^2$	$(34.06)^2$	$(34.06)^2$

FIGURE 8. Comparison of the exact value of fair variance, K_{var}, with the approximate value from the formula of Equation 33, as a function of the skew slope b. The thin line with squares shows the exact values obtained by replicating the log-payoff. The thick line depicts the approximate value given by Equation 33. (a) Three-month variance swap. (b) One-year variance swap.



PRACTICAL PROBLEMS WITH REPLICATION

We have shown in Equation 20 that a variance swap is theoretically equivalent to a dynamically adjusted, constant-dollar exposure to the stock, together with a static long position in a portfolio of options and a forward that together replicate the payoff of a log contract. This portfolio strategy captures variance exactly, provided the portfolio of options contains all strikes between zero and infinity in the appropriate weight to match the log payoff, and provided the stock price evolves continuously.

Two obvious things can go wrong. First, you may be able to trade only a limited range of options strikes, insufficient to accurately replicate the log payoff. Second, the stock price may jump. Both of these effects cause the strategy to capture a quantity that is not the true realized variance. We will focus on the effects of these two limitations below, though other practical issues, like liquidity, may also corrupt the ideal strategy.

Imperfect Replication Due to Limited Strike Range

Variance replication requires a log contract. Since log contracts are not traded in practice, we replicate the payoff with traded standard options in a limited strike range. Because these strikes fail to duplicate the log contract exactly, they will capture less than the true realized variance. Therefore, they have lower value than that of a true log contract, and so produce an inaccurate, lower estimate of the fair variance.

In Table 4 below we show how the estimated value of fair variance is affected by the range of strikes that make up the replicating portfolio. The fair variances are estimated from (1) a replicating portfolio with a narrow range of strikes, ranging from 75% to 125% of the initial spot level, and (2) a portfolio with a wide range of strikes, from 50% to 200% of the initial spot level. In both cases the strikes are uniformly spaced, one point apart. (The fair variance is calculated according to Equation 26, except that the integrals are replaced by sums over the available option strikes whose weights are chosen according to the procedure of Appendix A). We assume here that implied volatility is 25% per year for all strikes, with no volatility skew, so that all options are valued at the same implied volatility. We also assume a continuously compounded annual interest rate of 5%.

For both expirations, the wide strike range accurately approximates the actual square of the implied volatility. However, the narrow strike range underestimates the fair variance, more dramatically so for longer expirations.

TABLE 4. The effect of strike range on estimated fair variance.

Expiration	Wide strike range (50% - 200%)	Narrow strike range (75% - 125%)
Three-month	$(25.0)^2$	$(24.9)^2$
One-year	$(25.0)^2$	$(23.0)^2$

In the section entitled Replicating Variance Swaps: First Steps on page 6, we have already discussed one approach to understanding why the narrow strike range fails to capture variance. As shown in Figure 3, the vega and gamma of a limited strike range both fall to zero when the index moves outside the strike range, and the strategy then fails to accrue realized variance as the stock price moves. Consequently, the estimated variance is lower than the true fair value for both expirations above, and the reduction in value is greater for the one-year case. Over a longer time period it is more likely that the stock price will evolve outside the strike range.

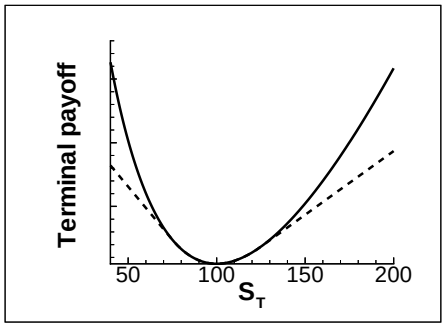
In essence, capturing variance requires owning the full log contract, whose duplication demands an infinite range of strikes. If you own a limited number of strikes, still appropriately weighted, you pay less than the full value, and, when the stock price evolves into regions where the curvature of the portfolio is insufficiently large, you capture less than the full realized variance, even if no jumps occur and the stock always moves continuously. In order to keep capturing variance, you need to maintain the curvature of the log contract at the current stock price, whatever value it takes.

A simpler way of understanding why a narrow strike range leads to a lower fair variance is to compare the payoff of the narrow-strike replicating portfolio at expiration to the terminal payoff that the portfolio is attempting to replicate, that is, the nonlinear part of the log payoff:

$$\frac{S_T - S_0}{S_0} - \log\frac{S_T}{S_0} \qquad \textbf{(EQ 34)}$$

Figure 9 displays the mismatch between the two payoffs. The narrow-strike option portfolio matches the curved part of the log payoff well at stock price levels between the range of strikes, that is, from 75 to 125. Beyond this range, the option portfolio payoff remains linear, always growing less rapidly than the nonlinear part of the log contract. The lack of curvature (or gamma, or vega) in the options portfolio outside

FIGURE 9. Comparison of the terminal payoff of the narrow-strike replicating portfolio (dashed line) and the nonlinear part of the log-payoff (solid line).



the narrow strike range is responsible for the inability to capture variance.

The Effect of Jumps on a Perfectly Replicated Log Contract

When the stock price jumps, the log contract may no longer capture realized volatility, for two reasons. First, if the log contract has been approximately replicated by only a finite range of strikes, a large jump may take the stock price into a region in which variance does not accrue at the right rate. Second, even with perfect replication, a discontinuous stock-price jump causes the variance-capture strategy of Equation 20 to capture an amount not equal to the true realized variance. In reality, both these effects contribute to the replication error. In this section, we focus only on the second effect and examine the effects of jumps assuming that the log-payoff can be replicated perfectly with options.

For the sake of discussion, from now on we will assume that we are short the variance swap, which we will hedge by following a discrete version of the variance-capture strategy

$$V = \frac{2}{T}\left[\sum_{i=1}^{N}\frac{\Delta S_i}{S_{i-1}} - \log\frac{S_T}{S_0}\right] \qquad \text{(EQ 35)}$$

where $\Delta S_i = S_i - S_{i-1}$ is the change in stock price between successive observations. Rather than continuously rebalance as the stock price moves, we instead adjust the exposure to $(2/T)$ dollars worth of stock only when a new stock price is recorded for updating the realized variance.

Because of the additive properties of the logarithm function, the terminal log payoff is equivalent to a daily accumulation of log payoffs:

$$V = \frac{2}{T}\sum_{i=1}^{N}\left[\frac{\Delta S_i}{S_{i-1}} - \log\frac{S_i}{S_{i-1}}\right] \quad \textbf{(EQ 36)}$$

Suppose that all but one of the daily price changes are well-behaved – that is, all changes are diffusive, except for a single jump event. We characterize the jump by the parameter J, the percentage jump *downwards*, from $S \to S(1-J)$; a jump downwards of 10% corresponds to J = 0.1. A jump up corresponds to a value $J < 0$.

The contribution of this one jump to the variance is easy to isolate, because variance is additive; the total (un-annualized) realized variance for a zero-mean contract is the sum

$$V = \frac{1}{T}\sum\left(\frac{\Delta S_i}{S_{i-1}}\right)^2 = \frac{1}{T}\sum_{\text{no jumps}}\left(\frac{\Delta S_i}{S_{i-1}}\right)^2 + \frac{1}{T}\left(\frac{\Delta S}{S}\right)^2_{\text{jump}} \quad \textbf{(EQ 37)}$$

The contribution of the jump to the realized total variance is given by:

$$\frac{1}{T}\left(\frac{\Delta S}{S}\right)^2_{\text{jump}} = \frac{J^2}{T} \quad \textbf{(EQ 38)}$$

On the other hand, the impact of the jump on the quantity captured by our variance replication strategy in Equation 36 is

$$\frac{2}{T}\left(\frac{\Delta S_i}{S_{i-1}} - \log\frac{S_i}{S_{i-1}}\right)_{\text{jump}} = \frac{2}{T}[-J - \log(1-J)] \quad \textbf{(EQ 39)}$$

In the limit that the jump size J is small enough to be regarded as part of a continuous stock evolution process, the right hand side of Equation 39 does reduce to the contribution of this (now small) move to the true realized variance. It is only because J is not small that the variance capture strategy is inaccurate. Therefore, the replication error, or the P&L (profit/loss) due to the jump for a short position in a variance swap hedged by a long position in a variance-capture strategy is

$$\text{P\&L due to jump} = \frac{2}{T}[-J - \log(1-J)] - \frac{J^2}{T} \quad \textbf{(EQ 40)}$$

To understand this result better, it is helpful to expand the log function as a series in J:

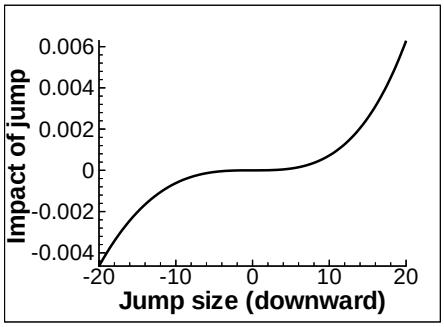
$$-\log(1-J) = J + \frac{J^2}{2} + \frac{J^3}{3} + \ldots \quad \textbf{(EQ 41)}$$

The leading contribution to the replication error is then

$$\text{P\&L due to jump} = \frac{2}{3}\frac{J^3}{T} + \ldots \qquad \textbf{(EQ 42)}$$

The quadratic contribution of the jump is the same for the variance swap as it is for the variance-capture strategy, and has no impact on the hedging mismatch. The leading correction is cubic in the jump size J and has a different sign for upwards or downwards jumps. A large move downwards ($J > 0$) leads to a profit for the (short variance swap)-(long variance-capture strategy), while a large move upwards ($J < 0$) leads to a loss. Furthermore, a large move one day, followed by a large move in the opposite direction the next day would tend to offset each other. Figure 10 shows the impact of the jump on the strategy for a range of jump values. Note that the simple cubic approximation of Equation 42 correctly predicts the sign of the P&L for all values of the jump size.

FIGURE 10. he impact of a single jump on the profit or loss of a short position in a variance swap and a long position in the variance replication strategy, as given by Equation 40 as a function of (downward) jump size for T=1 year.



There is an analogy between the cancellation of the quadratic jump term in variance replication and the linear jump term in options replication. When you are long an option you are long linear, quadratic and higher-order dependence on the stock price. If you are also short the option's delta-hedge, then the linear dependence of the net position cancels, leaving only the quadratic and higher-order dependencies. Because the leading-order term is quadratic, large moves in either direction benefit the position; this is precisely why hedged long options positions capture variance. In contrast, in the case of variance replication considered here, the variance replication strategy is long quadratic, cubic and higher-order terms in the stock price, while the position in the variance swap is short only the quadratic dependence. Now the quadratic term in the net position cancels, leaving only cubic and higher-order dependencies on the jump size. Since the leading

term is cubic, the direction of the jump determines whether there is a net profit or loss.

Table 5 displays the profit or loss due to jumps of varying sizes for three-month and one-year variance swaps.

TABLE 5. The profit/loss due to a single jump for a short variance swap with a notional value of $1 per squared variance point, that is hedged with the variance replication strategy of Equation 36 for T=1 year.

Jump size and direction	Three-month	One-year
J = 15% (down)	101.5	25.4
J = 10% (down)	28.8	7.2
J = 5% (down)	3.5	0.9
J = –5% (up)	–3.2	–0.8
J = –10% (up)	–24.8	–6.2
J = –20% (up)	–80.9	–20.2

The Effect of Jumps When Replicating With a Finite Strike Range

In practice, both the effects of jumps and the risks of log replication with only a limited strike range cause the strategy to capture a quantity different from the true realized variance of the stock price. The combined effect of both these risks is harder to characterize because they interact with one another in a complicated manner.

Consider again a short position in a variance contract that is being hedged by the variance-capture strategy. Suppose that a downward jump occurs, large enough to move the stock price outside the range of option strikes. If the log-payoff were replicated perfectly, the constant-dollar exposure would cancel the linear part of the stock price change, and lead to a convexity gain. Although the log-payoff is not being replicated perfectly, there is still a convexity gain from the jump, but it is smaller in size. However, after this jump, with the stock price now outside the strike range, the vega and gamma of the replicating portfolio are now too low to accrue sufficient variance, even if no further jumps occur. In this scenario, the gain from the jump has to be balanced against the subsequent failure of the hedge to capture the smooth variance. The net results will depend on the details of the scenario.

In contrast, a large move upwards will be doubly damaging: there will be convexity loss due to the jump and the hedge will not capture variance if the jump takes the index outside the strike range.

FROM VARIANCE TO VOLATILITY CONTRACTS

For most of this note we have focused on valuing and replicating variance swaps. But most market participants prefer to quote levels of volatility rather than variance, and so we now consider volatility swaps.

There is no simple replication strategy for synthesizing a volatility swap; it is variance that emerges naturally from hedged options trading. The replication strategy for the variance swap makes no assumptions about the level of future volatility, other than assuming that the stock price evolves continuously (without jumps). Changes in volatility have no effect on the strategy, which still captures the total variance over the life of the log contract. In contrast, as we will show, the replication strategy for a volatility swap is fundamentally different; it is affected by changes in volatility and its value depends on the volatility of future realized volatility. In essence, from a contingent claims or derivatives point of view, variance is the primary underlyer and all other volatility payoffs, such as volatility swaps, are best regarded as derivative securities on the variance as underlyer. From this perspective, volatility itself is a nonlinear function (the square root) of variance and is therefore more difficult, both theoretically and practically, to value and hedge.

To illustrate the issues involved, let's consider a naive strategy: approximate a volatility swap by statically holding a suitably chosen variance contract. In order to approximate a volatility swap struck at K_{vol}, which has payoff $\sigma_R - K_{vol}$, we can use the approximation

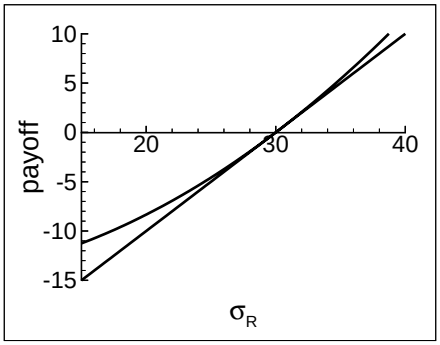
$$\sigma_R - K_{vol} \approx \frac{1}{2K_{vol}}(\sigma_R^2 - K_{vol}^2) \qquad \textbf{(EQ 43)}$$

This means that $1/(2K_{vol})$ variance contracts with strike K_{vol}^2 can approximate a volatility swap with a notional \$1/(vol point), for realized volatilities near K_{vol}. With this choice, the variance and volatility payoffs agree in value and volatility sensitivity (the first derivative with respect to σ_R) when $\sigma_R = K_{vol}$. Naively, this would also imply that the fair price of future volatility (the strike for which the volatility swap has zero value) is simply the square root of fair variance K_{var}:

$$K_{vol} = \sqrt{K_{var}} \qquad \text{(naive estimate)} \qquad \textbf{(EQ 44)}$$

In Figure 11 we compare the two sides of Equation 43 for K_{vol} = 30% for different values of the realized volatility. We see that the actual volatility swap and the approximating variance swap differ appreciably

FIGURE 11. Payoff of a volatility swap (straight line) and variance swap (curved line) as a function of realized volatility, for $K_{vol} = 30\%$.



only when the future realized volatility moves away from K_{vol}; you cannot fit a line everywhere with a parabola.

The naive estimate of Equations 43 and 44 is not quite correct. With this choice, the variance swap payoff is always greater than the volatility swap payoff. The mismatch between the variance and volatility swap payoffs in Equation 43, is the

$$\text{convexity bias} = \frac{1}{2K_{vol}}(\sigma_R - K_{vol})^2$$

This square is always positive, so that with this choice of the fair delivery price for volatility, the variance swap always outperforms the volatility swap. To avoid this arbitrage, we should correct our naive estimate to make the fair strike for the volatility contract lower than the square root of the fair strike for a variance contract, so that $K_{vol} < \sqrt{K_{var}}$. In this way, the straight line in Figure 11 will shift to the left and will not always lie below the parabola.

In order to estimate the size of the convexity bias, and therefore the fair strike for the volatility swap, it is necessary to make an assumption about both the level and volatility of future realized volatility. In Appendix D we estimate the expected hedging mismatch and static hedging parameters under the assumption that future realized volatility is normally distributed.

Dynamic Replication of a Volatility Swap

In principle, some of the risks inherent in the static approximation of a volatility swap by a variance swap could be reduced by dynamically trading new variance contracts throughout the life of the volatility swap. This dynamic replication of a volatility swap by means of vari-

ance swaps would (in principle) produce the payoff of a volatility swap independent of the moves in future volatility. This is closely analogous to replicating a curved stock option payoff by means of delta-hedging using the linear underlying stock price. In practice, of course, there is no market in variance swaps liquid enough to provide a usable underlyer.

In the same way that the appropriate option hedge ratio depends on the assumed future volatility of the stock, the dynamic replication of a volatility swap requires a model for the volatility of volatility. Taking the analogy further, one could imagine that the strategy would call for holding at every instant a "variance-delta" equivalent of variance contracts to hedge a volatility derivative.

The practical implementation of these ideas requires an arbitrage-free model for the stochastic evolution of the volatility surface. Due to the complexity of the mathematics involved, it is only very recently that such models have been developed [see for example Derman and Kani (1998) and Ledoit (1998)]. When there is a liquid market in variance swaps, these models may be useful in hedging volatility swaps and other variance derivatives.

CONCLUSIONS AND FUTURE INNOVATIONS

We have tried to present a comprehensive and didactic account of both the principles and methods used to value and hedge variance swaps. We have explained both the intuitive and the rigorous approach to replication. In markets with a volatility skew (the real world for most swaps of interest), the intuitive approach loses its footing. Here, using the rigorous approach, one can still value variance swaps by replication. Remarkably, we have succeeded in deriving analytic approximations that work well for the swap value under commonly used skew parameterizations. These formulas enable traders to update price quotes quickly as the market skew changes.

There are at least two areas where further development is welcome.

First, our ability to effectively price and hedge volatility swaps is still limited. To fully implement a replication strategy for volatility swaps, we need a consistent stochastic volatility model for options. Much work remains to be done in this area.

Second, some market participants prefer to enter a capped variance swap or volatility swap that limits the possible loss on the position. The capped variance swap has embedded in it an option on realized variance. The development of a truly liquid market in volatility swaps, forwards or futures would lead to the possibility of trading and hedging volatility options. Once again, this requires a consistent model for stochastic volatility.

APPENDIX A: REPLICATING LOGARITHMIC PAYOFFS

In this Appendix we derive several results concerning the replication of a logarithmic payoff with portfolios of standard options.

Constant Vega Requires Options Weighted Inversely Proportional to the Square of the Strike

Consider a portfolio of standard options

$$\Pi(S) = \int_0^\infty \rho(K) O(S, K, v)\, dK \qquad \textbf{(A 1)}$$

where $O(S,K,v)$ represents a standard Black-Scholes option of strike K and total variance $v = \sigma^2\tau$ when the stock price is S.

Vega, the sensitivity to the total variance of an individual option O in this portfolio, is given by

$$V_O = \tau \frac{\partial}{\partial v}(O) = \tau S f\left(\frac{K}{S}, v\right)$$

where

$$f(S, K, v) = \frac{1}{2\sqrt{v}} \frac{\exp(-d_1^2/2)}{\sqrt{2\pi}}$$

and

$$d_1 = \frac{\ln(S/K) + v/2}{\sqrt{v}}.$$

The variance sensitivity of the whole portfolio is therefore

$$V_\Pi(S) = \tau \int_0^\infty \rho(K) S f\left(\frac{K}{S}, v\right) dK \qquad \textbf{(A 2)}$$

The sensitivity of vega to S is

$$\begin{aligned} \frac{\partial V_\Pi}{\partial S} &= \tau \int_0^\infty \frac{\partial}{\partial S}[S^2 \rho(xS)] f(x, v)\, dx \\ &= \tau \int_0^\infty S[2\rho(xS) + xS\rho'(xS)] f(x, v)\, dx \end{aligned}$$

where, in the second line of the above equation, we changed the integration variable to $x = K/S$.

We want vega to be independent of S, that is $\frac{\partial V}{\partial S}_{\Pi} = 0$, which implies that

$$2\rho + K\frac{\partial \rho}{\partial K} = 0$$

The solution to this equation is

$$\rho = \frac{\text{const}}{K^2} \qquad \textbf{(A 3)}$$

Log Payoff Replication with a Discrete Set of Options

It was shown in the main text that the realized variance is related to trading a log contract. Since there is no log-contract traded, we want to represent it in terms of standard options. It is useful to subtract the linear part (corresponding to the forward contract) and look at the function

$$f(S_T) = \frac{2}{T}\left[\frac{S_T - S_*}{S_*} - \log\frac{S_T}{S_*}\right] \qquad \textbf{(A 4)}$$

where S_* is some reference price. In practice, only a discrete set of option strikes is available for replicating $f(S_T)$, and we need to determine the number of options for each strike. Assume that you can trade call options with strikes

$$K_0 = S_* < K_{1c} < K_{2c} < K_{3c} < \ldots$$

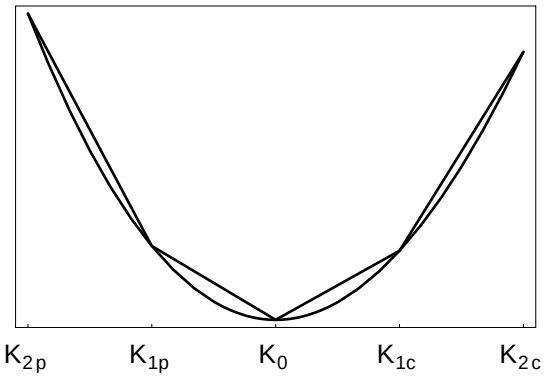
and put options with strikes

$$K_0 = S_* > K_{1p} > K_{2p} > K_{3p} > \ldots$$

We can approximate $f(S_T)$ with a piece-wise linear function as in Figure 11. The first segment to the right of S_* is equivalent to the payoff of a call option with strike K_0. The number of options is determined by the slope of this segment:

$$w_c(K_0) = \frac{f(K_{1c}) - f(K_0)}{K_{1c} - K_0} \qquad \textbf{(A 5)}$$

FIGURE 12. Log-payoff and options portfolio at maturity.



Similarly, the second segment looks like a combination of calls with strikes K_0 and K_{1c}. Given that we already hold $w_c(K_0)$ options with strike K_0 we need to hold $w_c(K_1)$ calls with strike K_1 where

$$w_c(K_1) = \frac{f(K_{2c}) - f(K_{1c})}{K_{2c} - K_{1c}} - w_c(K_0) \qquad \textbf{(A 6)}$$

Continuing in this way we can build the entire payoff curve one step at the time. In general, the number of call options of strike $K_{n,c}$ is given by

$$w_c(K_{n,c}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n+1,c} - K_{n,c}} - \sum_{i=0}^{n-1} w_c(K_{i,c}) \qquad \textbf{(A 7)}$$

The other side of the curve can be built using put options:

$$w_p(K_{n,p}) = \frac{f(K_{n+1,p}) - f(K_{n,p})}{K_{n,p} - K_{n+1,p}} - \sum_{i=0}^{n-1} w_c(K_{i,p}) \qquad \textbf{(A 8)}$$

APPENDIX B: SKEW LINEAR IN STRIKE

Here, we derive a formula which gives the approximate value of the variance swap when the skew is linear in strike. We parameterize the implied volatility by

$$\Sigma(K) = \Sigma_0 - b\frac{K - S_F}{S_F} \tag{B 1}$$

where $S_F = S_0 e^{rT}$ is the forward value corresponding to the current spot, Σ_0 is at-the-money forward implied volatility and b is the slope of the skew.

We start with the general expression for the fair variance discussed in the main text:

$$K_{var} = \frac{2}{T}\left(rT - \left(\frac{S_0}{S_*}e^{rT} - 1\right) - \log\frac{S_*}{S_0} + e^{rT}\int_0^{S_*}\frac{1}{K^2}P(K,\Sigma(b))dK + e^{rT}\int_{S_*}^{\infty}\frac{1}{K^2}C(K,\Sigma(b))dK\right) \tag{B 2}$$

We now expand option prices as a power series in b around a flat implied volatility ($b = 0$),

$$\begin{aligned} C(K,\Sigma(b)) &= C(K,\Sigma_0) + b\frac{\partial C}{\partial b}\bigg|_{b=0} + \frac{1}{2}b^2\frac{\partial^2 C}{\partial b^2}\bigg|_{b=0} + \ldots \\ P(K,\Sigma(b)) &= P(K,\Sigma_0) + b\frac{\partial P}{\partial b}\bigg|_{b=0} + \frac{1}{2}b^2\frac{\partial^2 P}{\partial b^2}\bigg|_{b=0} + \ldots \end{aligned} \tag{B 3}$$

Using this expansion we can formally write an expansion of fair variance in powers of b as follows:

$$\begin{aligned} K_{var} = \Sigma_0^2 &+ b\left(\frac{2}{T}e^{rT}\right)\left\{\int_0^{S_*}\frac{1}{K^2}\frac{\partial P}{\partial b}\bigg|_{b=0}dK + \int_{S_*}^{\infty}\frac{1}{K^2}\frac{\partial C}{\partial b}\bigg|_{b=0}dK\right\} + \\ &\frac{1}{2}b^2\left(\frac{2}{T}e^{rT}\right)\left\{\int_0^{S_*}\frac{1}{K^2}\frac{\partial^2 P}{\partial b^2}\bigg|_{b=0}dK + \int_{S_*}^{\infty}\frac{1}{K^2}\frac{\partial^2 C}{\partial b^2}\bigg|_{b=0}dK\right\} + \ldots \end{aligned} \tag{B 4}$$

Here Σ_0^2 is the fair variance in the "flat world" where volatility is constant and is given by Equation B2 with $\Sigma(b)$ replaced by Σ_0.

The derivatives which enter Equation B4 are given by

$$\left.\frac{\partial P}{\partial b}\right|_{b=0} = \left.\frac{\partial P}{\partial \Sigma}\right|_{\Sigma_0} \left.\frac{\partial \Sigma}{\partial b}\right|_{b=0}, \qquad \left.\frac{\partial^2 P}{\partial b^2}\right|_{b=0} = \left.\frac{\partial^2 P}{\partial \Sigma^2}\right|_{\Sigma_0} \left.\left(\frac{\partial \Sigma}{\partial b}\right)^2\right|_{b=0}$$

$$\left.\frac{\partial C}{\partial b}\right|_{b=0} = \left.\frac{\partial C}{\partial \Sigma}\right|_{\Sigma_0} \left.\frac{\partial \Sigma}{\partial b}\right|_{b=0}, \qquad \left.\frac{\partial^2 C}{\partial b^2}\right|_{b=0} = \left.\frac{\partial^2 C}{\partial \Sigma^2}\right|_{\Sigma_0} \left.\left(\frac{\partial \Sigma}{\partial b}\right)^2\right|_{b=0}$$

The derivatives with respect to volatility are easily calculated using the Black-Scholes formula

$$\begin{aligned} \left.\frac{\partial P}{\partial \Sigma}\right|_{\Sigma_0} &= \left.\frac{\partial C}{\partial \Sigma}\right|_{\Sigma_0} = \frac{S\sqrt{T}}{\sqrt{2\pi}} e^{-d_1^2/2} \\ \left.\frac{\partial^2 P}{\partial \Sigma^2}\right|_{\Sigma_0} &= \left.\frac{\partial^2 C}{\partial \Sigma^2}\right|_{\Sigma_0} = -\frac{S\sqrt{T}}{\sqrt{2\pi}} d_1 \frac{\partial d_1}{\partial \Sigma_0} e^{-d_1^2/2} \\ d_1 &= \frac{\log\left(\frac{S_F}{K}\right) + \frac{1}{2}\Sigma_0^2 T}{\Sigma_0\sqrt{T}} \end{aligned} \tag{B 5}$$

where, for the model we are considering here

$$\left.\frac{\partial \Sigma}{\partial b}\right|_{b=0} = -\left(\frac{K}{S_F} - 1\right) \tag{B 6}$$

The fact that call and put options have the same vega in the Black-Scholes framework makes it possible to combine the integrals in Equation B4 into one integral from 0 to ∞:

$$\begin{aligned} K_{var} = \Sigma_0^2 - b\left(\frac{2}{T}e^{rT}\right)\frac{S\sqrt{T}}{\sqrt{2\pi}}\int_0^\infty \frac{1}{K^2}\left(\frac{K}{S_F} - 1\right)e^{-d_1^2/2} dK - \\ \frac{1}{2}b^2\left(\frac{2}{T}e^{rT}\right)\frac{S\sqrt{T}}{\sqrt{2\pi}}\int_0^\infty \frac{1}{K^2}\left(\frac{K}{S_F} - 1\right)^2 d_1 \frac{\partial d_1}{\partial \Sigma_0} e^{-d_1^2/2} dK + \ldots \end{aligned} \tag{B 7}$$

To evaluate these integrals, one can, for example, change the integration variable to $z = \left(\log\frac{S_F}{K} + \frac{1}{2}v_0\right)/\sqrt{v_0} \equiv d_1$, where $v_0 = \Sigma_0^2 T$, and then write Equation B7 as

$$K_{var} = \Sigma_0^2 - b\left\{2\Sigma_0\int_{-\infty}^{\infty}\left(1 - e^{\sqrt{v_0}z - v_0/2}\right)e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}\right\} +$$

$$b^2\left\{\int_{-\infty}^{\infty}\left(e^{\sqrt{v_0}z - v_0/2} + e^{-\sqrt{v_0}z + v_0/2} - 2\right)(z^2 - \sqrt{v_0}z)e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}\right\}$$

The term linear in b vanishes and the term quadratic in b has coefficient $3\Sigma_0^2 T$, so that

$$K_{var} = \Sigma_0^2(1 + 3Tb^2 + ...) \qquad \textbf{(B 8)}$$

We now present an alternative derivation of this result. We start with the fundamental definition of the fair delivery variance as the expected value of future realized variance, i.e.

$$K_{var} = E\left[\frac{1}{T}\int_0^T \sigma^2(S,t)dt\right] \qquad \textbf{(B 9)}$$

This can be evaluated approximately as follows. First, we use the relation between implied and local volatility:

$$\sigma^2(S,t) = \left.\frac{\frac{\Sigma}{T} + 2\frac{\partial\Sigma}{\partial T} + 2rK\frac{\partial\Sigma}{\partial K}}{K^2\left\{\frac{\partial^2\Sigma}{\partial K^2} - d_1\sqrt{T}\left(\frac{\partial\Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left(\frac{1}{K\sqrt{T}} + d_1\frac{\partial\Sigma}{\partial K}\right)^2\right\}}\right|_{\substack{K = S \\ T = t}} \qquad \textbf{(B 10)}$$

Denote $x = \frac{S}{S_F} - 1$. Equation B10 can be written as

$$\sigma^2(S,t) = \frac{\Sigma_0 - bx - 2br(1+x)t}{(1+x)^2 t\left\{-b^2\sqrt{t}d_1 + \frac{1}{\Sigma_0 - bx}\left[\frac{1}{(1+x)\sqrt{t}} - bd_1\right]^2\right\}} \qquad \textbf{(B 11)}$$

where

$$d_1 = \frac{-\log(1+x)}{(\Sigma_0 - bx)\sqrt{t}} + \frac{1}{2}(\Sigma_0 - bx)\sqrt{t} \qquad \textbf{(B 12)}$$

We expand $\sigma^2(S,t)$ in powers of x and calculate the expected value in a lognormal world with volatility Σ_0 using

$$E[x] = 0$$
$$E[x^2] = e^{\Sigma_0^2 t} - 1$$
$$E[x^3] = e^{3\Sigma_0^2 t} - 3e^{\Sigma_0^2 t} + 2$$
$$\ldots$$

Expected values of higher powers of x are easily calculated using

$$E\left[\left(\frac{S}{S_F}\right)^n\right] = e^{(n^2-n)\Sigma_0^2 t/2}$$

After averaging over the stock price distribution, we average over time and, finally, expand the result in powers of the skew slope b. Tedious calculation leads to the relation

$$K_{var} = \Sigma_0^2(1 + 3Tb^2 + \ldots.)$$

It is reassuring that these two very different methods lead to the same approximation formula.

APPENDIX C: SKEW LINEAR IN DELTA

Here we consider the case where implied volatility varies linearly with delta. Such a skew can be parameterized in terms of Δ_p, the delta of a European-style put, as

$$\Sigma(\Delta_p) = \Sigma_0 + b\left(\Delta_p + \frac{1}{2}\right) \qquad \textbf{(C 1)}$$

where Σ_0 is the implied volatility of options with $\Delta_p = -1/2$ (the "50-delta volatility"). (We could also parametrize the skew in terms of the call delta as $\Sigma(\Delta_c) = \Sigma_0 + b\left(\Delta_c - \frac{1}{2}\right)$.)

To derive the formula for the fair variance we follow the same procedure as in Appendix B, starting with Equation B2. One important difference is that now implied volatility is nonlinear in b (since Δ_p depends implicitly on b) so that second derivatives have an additional term:

$$\begin{aligned}
\left.\frac{\partial^2 P}{\partial b^2}\right|_{b=0} &= \left.\frac{\partial^2 P}{\partial \Sigma^2}\right|_{\Sigma_0} \left.\left(\frac{\partial \Sigma}{\partial b}\right)^2\right|_{b=0} + \left.\frac{\partial P}{\partial \Sigma}\right|_{\Sigma_0} \left.\frac{\partial^2 \Sigma}{\partial b^2}\right|_{b=0} \\
\left.\frac{\partial^2 C}{\partial b^2}\right|_{b=0} &= \left.\frac{\partial^2 C}{\partial \Sigma^2}\right|_{\Sigma_0} \left.\left(\frac{\partial \Sigma}{\partial b}\right)^2\right|_{b=0} + \left.\frac{\partial C}{\partial \Sigma}\right|_{\Sigma_0} \left.\frac{\partial^2 \Sigma}{\partial b^2}\right|_{b=0}
\end{aligned} \qquad \textbf{(C 2)}$$

Other derivatives we need are easily calculated:

$$\begin{aligned}
\left.\frac{\partial \Sigma}{\partial b}\right|_{b=0} &= \Delta_p + \frac{1}{2} \\
\left.\frac{\partial^2 \Sigma}{\partial b^2}\right|_{b=0} &= 2\left(\Delta_p + \frac{1}{2}\right)\frac{\partial \Delta_p}{\partial \Sigma_0}
\end{aligned} \qquad \textbf{(C 3)}$$

where

$$\Delta_p = -N(-d_1)$$

and

$$d_1 = \frac{\log\frac{S_F}{K}}{\Sigma_0\sqrt{T}} + \frac{1}{2}\Sigma_0\sqrt{T}$$

Combining these relations, the fair variance can be written as

$$K_{var} = \Sigma_0^2 - b\left(\frac{2}{T}e^{rT}\right)\frac{S\sqrt{T}}{\sqrt{2\pi}}\int_0^\infty \frac{1}{K^2}\left(\Delta_p + \frac{1}{2}\right)e^{-d_1^2/2}dK -$$

$$\frac{1}{2}b^2\left(\frac{2}{T}e^{rT}\right)\frac{S\sqrt{T}}{\sqrt{2\pi}}\left(\int_0^\infty \frac{1}{K^2}\left(\Delta_p + \frac{1}{2}\right)^2 d_1\frac{\partial d_1}{\partial\Sigma_0}e^{-d_1^2/2}dK\right. \qquad \textbf{(C 4)}$$

$$\left. -2\int_0^\infty \frac{1}{K^2}\left(\Delta_p + \frac{1}{2}\right)\frac{\partial\Delta_p}{\partial\Sigma_0}e^{-d_1^2/2}dK\right)$$

Again, integrals can be evaluated by changing the integration variable to $z = \left(\log\frac{S_F}{K} + \frac{1}{2}v_0\right)/\sqrt{v_0} \equiv d_1$, where $v_0 = \Sigma_0^2 T$, so that

$$K_{var} = \Sigma_0^2 - b\left\{2\Sigma_0\int_{-\infty}^\infty \left[N(z) - \frac{1}{2}\right]e^{\sqrt{v_0}z - v_0/2}e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}\right\} +$$

$$b^2\left(\int_{-\infty}^\infty \left[N(z) - \frac{1}{2}\right]^2(z^2 - \sqrt{v_0}z)e^{\sqrt{v_0}z - v_0/2}e^{-z^2/2}\frac{dz}{\sqrt{2\pi}}\right.$$

$$\left. -2\int_{-\infty}^\infty \left[N(z) - \frac{1}{2}\right](z - \sqrt{v_0})e^{\sqrt{v_0}z - v_0/2}e^{-z^2}\frac{dz}{2\pi}\right)$$

All these integrals can be evaluated exactly. Since we are eventually interested in expanding the result in powers of $v_0 = \Sigma_0^2 T$, one can first expand $e^{\sqrt{v_0}z - v_0/2}$ in powers of z and integrate term by term. It is also useful to note that $N(z) - \frac{1}{2}$ is antisymmetric in z to simplify calculations. In addition the following results are useful:

$$\int_0^\infty z^{2n} e^{-az^2/2}\left[N(bz)-\frac{1}{2}\right]\frac{dz}{\sqrt{2\pi}} = \frac{(-2)^n}{2\pi}\frac{\partial^n}{\partial a^n}\left[\frac{1}{\sqrt{a}}\arctan\frac{b}{\sqrt{a}}\right]$$

$$\int_0^\infty z^{2n+1} e^{-az^2/2}\left[N(bz)-\frac{1}{2}\right]\frac{dz}{\sqrt{2\pi}} = \frac{(-2)^n b}{2\sqrt{2\pi}}\frac{\partial^n}{\partial a^n}\left[\frac{1}{a\sqrt{a+b^2}}\right]$$

$$\int_0^\infty z^{2n} e^{-az^2/2}\left[N(bz)-\frac{1}{2}\right]^2\frac{dz}{\sqrt{2\pi}} = \frac{(-2)^n}{2\pi}\frac{\partial^n}{\partial a^n}\left[\frac{1}{\sqrt{a}}\left(\arctan\frac{\sqrt{a+2b^2}}{\sqrt{a}}-\frac{\pi}{4}\right)\right] \qquad \textbf{(C 5)}$$

$$\int_0^\infty z^{2n+1} e^{-az^2/2}\left[N(bz)-\frac{1}{2}\right]^2\frac{dz}{\sqrt{2\pi}} = \frac{(-2)^n b}{(2\pi)^{3/2}}\frac{\partial^n}{\partial a^n}\left[\frac{1}{a\sqrt{a+b^2}}\left(\arctan\frac{b}{\sqrt{a+b^2}}\right)\right]$$

After evaluating all integrals we find the final answer to be

$$K_{var} = \Sigma_0^2 + b\Sigma_0^2\sqrt{\frac{T}{\pi}} + \frac{1}{12}b^2 + ... \qquad \textbf{(C 6)}$$

Two- Slope Model

Our calculations can easily be generalized to the model where the slope of the skew is different for put and call options, i.e.

$$\begin{aligned}\Sigma_p(\Delta_p) &= \Sigma_0 + b_p\left(\Delta_p+\frac{1}{2}\right) \quad \text{for} \quad -\frac{1}{2}\le\Delta_p\le 0\\ \Sigma_c(\Delta_c) &= \Sigma_0 + b_c\left(\Delta_c-\frac{1}{2}\right) \quad \text{for} \quad 0\le\Delta_c\le\frac{1}{2}\end{aligned} \qquad \textbf{(C 7)}$$

We now briefly sketch the derivation emphasizing only the differences with the above detailed calculations. We start with the same fundamental expression:

$$\begin{aligned}K_{var} = \frac{2}{T}\Bigg(rT-\left(\frac{S_0}{S_*}e^{rT}-1\right)-\log\frac{S_*}{S_0}+\\ e^{rT}\int_0^{S_*}\frac{1}{K^2}P(K,\Sigma_p(b_p))dK + e^{rT}\int_{S_*}^\infty\frac{1}{K^2}C(K,\Sigma_c(b_c))dK\Bigg)\end{aligned} \qquad \textbf{(C 8)}$$

We use different implied volatility parameterizations for put and call options, as given by Equation C7. Note that we should choose S_* so that

$$S_* = S_F e^{-\Sigma_0^2 T/2}$$

This ensures that we use the put (call) parameterization in Equation C7 for strikes below (above) S_*. We expand put option prices in powers of b_p and call option prices in powers of b_c. Evaluating all integrals as above we find

$$K_{var} = \Sigma_0^2 + \left[\frac{1}{4}\Sigma_0(b_p - b_c) + \Sigma_0 \frac{b_p + b_c}{2}\left(\Sigma_0\sqrt{\frac{T}{\pi}}\right)\right] + \frac{1}{12}\frac{b_p^2 + b_c^2}{2} + ... \quad \textbf{(C 9)}$$

Obviously, for $b_p = b_c$ this reduces to the result for single slope given in Equation C6. Note that by changing the sign of b_c we turn the implied skew into a smile.

APPENDIX D: STATIC AND DYNAMIC REPLICATION OF A VOLATILITY SWAP

We have argued that volatility swaps are fundamentally different from variance swaps and that, unlike the variance swap, there is no simple replicating strategy to synthetically create a volatility swap.

In the section From Variance to Volatility Contracts on page 33, we showed that attempting to create a volatility swap from a variance swap by means of a "buy-and-hold" strategy invariably leads to misreplication, since this amounts to trying to fit a linear payoff (the volatility payoff) with a quadratic payoff (the variance swap).

Given a view on both the direction and volatility of future volatility, we will show that it is possible to pick the strike and notional size of a variance contract to match the payoff of a volatility contract, on average, as closely as possible. The extent of the replication mismatch will depend on how close the realized volatility is to its expected value.

The hedging instrument is the realized variance (Σ_T^2), while the target of the replication is the realized volatility (Σ_T). We want to approximate the volatility as a function of the variance by writing

$$\Sigma_T \approx a\Sigma_T^2 + b \qquad \textbf{(D 1)}$$

and choose a and b to minimize the expected squared deviation of the two sides of Equation D1:

$$\min\ E[(\Sigma_T - a\Sigma_T^2 - b)^2] \qquad \textbf{(D 2)}$$

Differentiation leads to the following equations for the coefficients a and b:

$$\begin{aligned} E[\Sigma_T] &= aE[\Sigma_T^2] + b \qquad \textbf{(D 3)} \\ E[\Sigma_T^3] &= aE[\Sigma_T^4] + bE[\Sigma_T^2] \end{aligned}$$

The distribution of future volatility could be assumed to be normal, with mean $\bar{\Sigma}$ and standard deviation σ_Σ:

$$\Sigma_T \sim N(\bar{\Sigma}, \sigma_\Sigma) \qquad \textbf{(D 4)}$$

This model only makes sense if the probability of negative volatilities is negligible. This strategy will replicate only on average; the expected squared replication error is given by:

$$\min E[(\Sigma_T - a\Sigma_T^2 - b)^2] = \text{Var}(\Sigma_T)[1 - (\text{corr}(\Sigma_T, \Sigma_T^2))^2] \quad \textbf{(D 5)}$$

For realized volatilities distributed normally as in Equation D4, the hedging coefficients are

$$a = \frac{1}{2\bar{\Sigma} + \frac{\sigma_\Sigma^2}{\bar{\Sigma}}} \quad \textbf{(D 6)}$$

$$b = \frac{\bar{\Sigma}}{2 + \frac{\sigma_\Sigma^2}{\bar{\Sigma}^2}}$$

and the expected squared replication error is:

$$\min E[(\Sigma_T - a\Sigma_T^2 - b)^2] = \frac{\sigma_\Sigma^2}{1 + \frac{2\bar{\Sigma}^2}{\sigma_\Sigma^2}} \quad \textbf{(D 7)}$$

REFERENCES

Carr, P. and D. Madan (1998). Towards a Theory of Volatility Trading, in Volatility: New Estimation Techniques for Pricing Derivatives, edited by R. Jarrow, 417-427.

Derman, E. and I. Kani (1994). Riding on a Smile, *RISK 7, No. 2*, 32-39.

Derman, E., I. Kani and J. Zou (1996). The Local Volatility Surface, *Financial Analyst Journal, July/August*, 25-36.

Derman, E., M. Kamal, I. Kani, and J. Zou (1996). *Valuing Contracts with Payoffs Based on Realized Volatility*, Global Derivatives (July) Quarterly Review, Equity Derivatives Research, Goldman, Sachs & Co.

Derman, E., M. Kamal, I. Kani, J. McClure, C. Pirasteh, and J. Zou (1996). *Investing in Volatility*, Quantitative Strategies Research Notes, October, Goldman, Sachs & Co.

Derman, E. and I. Kani (1998). Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility, *International Journal of Theoretical and Applied Finance, Vol. 1, No. 1*, 61-110.

Dupire, B. (1994). Pricing with a Smile, *RISK 7, No. 1*, 18-20.

Ledoit, O. and P. Santa-Clara (1998). Relative Price of Options with Stochastic Volatility. Working Paper, The Anderson Graduate Schools of Management, University of California, Los Angeles.

Neuberger, A. (1994). The Log Contract: A new instrument to hedge volatility, *Journal of Portfolio Management, Winter*, 74-80.

Neuberger, A. (1996). The Log Contract and Other Power Contracts, in The Handbook of Exotic Options, edited by I. Nelken, 200-212.

SELECTED QUANTITATIVE STRATEGIES PUBLICATIONS

June 1990 *Understanding Guaranteed Exchange-Rate Contracts In Foreign Stock Investments*
Emanuel Derman, Piotr Karasinski and Jeffrey Wecker

Jan. 1992 *Valuing and Hedging Outperformance Options*
Emanuel Derman

Mar. 1992 *Pay-On-Exercise Options*
Emanuel Derman and Iraj Kani

June 1993 *The Ins and Outs of Barrier Options*
Emanuel Derman and Iraj Kani

Jan. 1994 *The Volatility Smile and Its Implied Tree*
Emanuel Derman and Iraj Kani

May 1994 *Static Options Replication*
Emanuel Derman, Deniz Ergener and Iraj Kani

May 1995 *Enhanced Numerical Methods for Options with Barriers*
Emanuel Derman, Iraj Kani, Deniz Ergener and Indrajit Bardhan

Dec. 1995 *The Local Volatility Surface: Unlocking the Information in Index Option Prices*
Emanuel Derman, Iraj Kani and Joseph Z. Zou

Feb. 1996 *Implied Trinomial Trees of the Volatility Smile*
Emanuel Derman, Iraj Kani and Neil Chriss

Apr. 1996 *Model Risk*
Emanuel Derman,

Aug. 1996 *Trading and Hedging Local Volatility*
Iraj Kani, Emanuel Derman and Michael Kamal

Oct. 1996 *Investing in Volatility*
Emanuel Derman, Michael Kamal, Iraj Kani, John McClure, Cyrus Pirasteh and Joseph Zou

Apr. 1997	*Is the Volatility Skew Fair?*	Emanuel Derman, Michael Kamal, Iraj Kani and Joseph Zou
Apr. 1997	*Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility*	Emanuel Derman and Iraj Kani
Sept. 1997	*The Patterns of Change in Implied Index Volatilities*	Michael Kamal and Emanuel Derman
Nov. 1997	*Predicting the Response of Implied Volatility to Large Index Moves: An October 1997 S&P Case Study*	Emanuel Derman and Joe Zou
Sept. 1998	*How to Value and Hedge Options on Foreign Indexes*	Kresimir Demeterfi
Jan. 1999	*Regimes of Volatility: Some Observations on the Variation of S&P 500 Implied Volatilities*	Emanuel Derman