Outperformance Options

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To me, outperformance options[1] are especially interesting because of the alternating layers of complexity and simplicity you discover as you probe more deeply into their valuation.

This note describes a journey through these layers. First we describes the outperformance option's payoff, a simple function of two underlyers. But valuing a two-underlyer options seems complicated; each underlyer has to be hedged. It turns out that, if you think about it the right way, a general principle lets you find the correct and simple formula for its value. Even an options novice, who understands only single-underlyer options and the Black-Scholes formula, already knows everything necessary to value it, European- or American-style. We then examine the simple formulat for the outperformance option value, and notice a surprising, elegant and at first puzzling relation among its two hedge ratios. Finally, we explain the origin of this result.

The Payoff

Suppose you are a dollar-based investor with a choice between two stocks, *A* and *B*, with an investment horizon from today out to time *T* from today. Call *A(t)* and *B(t)* the unknown future dollar values at some future time *t* of a one-dollar investment today in stock *A* and stock *B* respec-

1. The Value of An Option to Exchange One Asset for Another, William Margrabe, Journal of Finance, Vol XXXIII no. 1, 177-186, March 1978.

tively, so that $A(0) = B(0) = \$1$. A European-style outperformance option on *A vs B* with time *T* to expiration and face value of $1 is a contract whose payoff in dollars at time T is equal to

$$\max [A(T) - B(T), 0] \qquad \text{(EQ 1)}$$

If you own just one of these options, you will obtain any positive excess return of *A* over *B* at time *T*. Owning 100 of them is equivalent to owning an outperformance option with a face value of $100.

An American-style outperformance option allows the holder of the contract to exercise early and receive the payoff of Equation 1 at any time *t* prior to expiration.

Most of this note will focus on European-style outperformance options, and spell out the simple extension to American-style outperformance options at the end.

The Financial Principle of Relativity of Currency

The outperformance option in Equation 1 is the right to exchange one share of stock *B* for one share of stock *A*. It looks complicated to value the option because it has exposure to *two underlyers,* and no-arbitrage pricing dictates that you will need to hedge it with two securities.

However, notice that an ordinary call option on stock *A* with strike of $1 has payoff *max [A(T) - $1, 0]*. It is the right to exchange one dollar for one share of stock *A*. If you can think of one share of stock *B* in Equation 1 as a sort of monetary unit, like one dollar, then the outperformance option resembles an ordinary call.

Here, we will show that you can clearly understand outperformance options by thinking of them as call options issued in an imaginary foreign country (*B-Land*) whose currency unit is one share of stock *B*. In *B-Land*, where all monetary values are denominated in shares of *B*, the outperformance option is simply a call option on a single stock *A* with a strike of one *B-share.* The value of the outperformance call in dollars is the value of this call (in *B-shares*), converted to dollars at the current *B-share*/dollar exchange rate. The relevant volatility of the call is the volatility of *A* in *B-shares*. To hedge the dollar value of the outperformance call, first you must hedge it against changes in the value of *A* in *B-Land*, and second, against changes in the the *B-share*/dollar exchange rate. The whole procedure is similar to the one you would follow as a dollar-based investor in a yen-denominated call on the Nikkei index, assuming you wanted to hedge against both changes in the index level and changes in the yen.

This method is sometimes called *the method of change of numeraire*. It's a kind of financial principle of currency relativity, which demands that the relative values of two securities should be independent of the currencies you use to express the value of each of them. It has many applications, and using it can remove the apparent complexity of many valuation problems.

Notation

We will need a notation that makes explicit the currency in which a security value is being quoted.

T	expiration time
t	any intermediate time between 0 and T
$S_i(t)$	value in currency i at time t of stock S that was worth $1 at time $t = 0$.
$\sigma(S_i)$	volatility of stock S in currency i
d_S	continuous dividend rate for stock S
$C_i^{AB}(t)$	value in currency i at time t of outperformance option with payoff of EQ 1.
$BS(S,K,r,\sigma,T-t)$	Black-Scholes formula for a stock with price S, strike K, continuous riskless rate r, return volatility σ, time to expiration $T-t$
Δ	the Black-Scholes hedge ratio $\frac{\partial}{\partial S}BS(\ )$

In this notation, Equation 1 says that an outperformance option has the dollar payoff at maturity

$$C_{\$}^{AB}(T) = max[A_{\$}(T) - B_{\$}(T), 0] \qquad \text{(EQ 2)}$$

The Outperformance Option's Value in B-Shares

Suppose you live in *B-Land* where the currency is the *B-share.* In this country, the value of one share of stock A in *B-shares* is $A_B(t) \equiv A_\$(t)/B_\$(t)$. The value of one *B-share* in dollars at t=0 was \$1. The value of one *B-share* in *B-Land* is $B_B(t)$ and is always equal to 1. The riskless interest rate at which you can earn interest on your *B-shares* is *B's* dividend rate d_B. The outperformance call in Equation 2 has a terminal payoff in *B-shares* given by dividing Equation 2 by $B_\$(T)$:

$$C_B^{AB}(T) = max[A_B(T) - 1, 0] \qquad \text{(EQ 3)}$$

This is simply the payoff at time *T* of an ordinary Black-Scholes European option on $A_B(t)$ with strike equal to 1 and volatility $\sigma(A_B)$ equal to the volatility of $A_\$(t)$ measured in *B-shares.* It is well known that

$$\sigma(A_B) = \sqrt{\sigma^2(A_\$) + \sigma^2(B_\$) - 2\rho_{AB}\sigma(A_\$)\sigma(B_\$)} \qquad \text{(EQ 4)}$$

where ρ_{AB} is the correlation between the returns of $A_\$(t)$ and $B_\$(t)$. The value at time t of this European-style call with payoff described by Equation 3 is therefore given by the familiar Black-Scholes formula

$$C_B^{AB}(t) = BS(A_B, 1, d_B, \sigma(A_B), T - t) \qquad \text{(EQ 5)}$$

The Outperformance Option's Value in Dollars

You now know the outperformance option's value in *B-shares*. You can get the value of the outperformance option in dollars at time *t* by taking its value in *B-shares* at time *t*, and converting it to dollars by multiplying by the exchange rate $B_\$(t)$, the value of a *B-share* in dollars at time *t*:

$$C_\$^{AB}(t) = B_\$(t) C_B^{AB}(t) = B_\$(t) BS(A_B, 1, d_B, \sigma(A_B), T-t) \qquad \text{(EQ 6)}$$

Hedging Outperformance Options

A long position in the outperformance option $C_B^{AB}(t)$ has exposure to the value of both *A*- and *B-shares*. The form of Equation 6 — *BS( )* depends only on the value A_B but not on $A_\$$ or $B_\$$ separately — suggests the natural way to think about hedging is in two steps. First hedge $BS(\ )$, the value of the call in *B-Land*, against changes in the value of A_B. Then hedge this partially hedged position against changes in the dollar value of a *B-share* to achieve a totally hedged position.

You can hedge the exposure to changes in $A_B(t)$ by going short Δ shares of A against the call. Δ is the usual Black-Scholes hedge ratio corresponding to Equation 5. It tells you how to hedge the call in *B-Land*. The value of this partially delta-hedged portfolio in *B-shares* is

$$P_B(t) = C_B^{AB}(t) - \Delta A_B(t) \qquad \text{(EQ 7)}$$

The Black-Scholes formula will guarantee that $P_B(t)$ is negative, so if you own the delta-hedged portfolio, you are instantaneously short $\Delta A_B(t) - C_B^{AB}(t)$ *B-shares*.

Your partially-hedged portfolio P_B is long one *B-share*-denominated call on *A* and short Δ shares of *A*. Its value in dollars is $P_B(t)B_\$(t)$. You have no exposure to the value of *A* denominated in *B-shares*. As a dollar investor, though, you are exposed to $B_\$(t)$, the *B-share*/dollar exchange rate. You can eliminate this residual exposure to *B-shares* by buying the *B-shares* you need to cancel the *B-share* exposure of the delta-hedged portfolio in Equation 7 - that is, by buying $\Delta A_B(t) - C_B^{AB}(t)$ units of *B-shares*, each worth $B_\$(t)$. *You have to hedge the premium against change in the B-share currency as well as the hedge.* Adding these to your portfolio makes it hedged against the instantaneous changes in value of either underlyer.

The final hedged portfolio, hedged against small moves in both $A_\$$ and $B_\$$, is shown in Table 1.

Table 1: The Constituents of the Totally Hedged Portfolio

Security	Hedge Quantity	Position Value (dollars)
Outperformance option	1	$C_B^{AB}(t)B_\$(t)$
A	$\Delta_A = -\Delta$	$-\Delta A_B(t)B_\$(t)$
B	$\Delta_B = \Delta A_B(t) - C_B^{AB}(t)$	$(\Delta A_B(t) - C_B^{AB}(t))B_\$(t)$
	Net position value:	0

Hedging and Scaling

Table 1 displays the *A-hedge* plus the *B-hedge* that together remove all exposure of the outperformance option to movements in *A* and *B*. Notice that the net position value of the totally hedged portfolio is zero.

There are two ways to interpret this. The first is to conclude that if you want a portfolio to be hedged against all securities, including its denomination currency, the portfolio must be worth zero.

The second is to notice that the outperformance option is homogeneous in the value of its underlying securities. You can write its value *C* at time t as

$$C(A, B, ..., t) = Bf(A/B, ..., \mathrm{t}) \qquad \text{(EQ 8)}$$

or

$$\frac{C(A, B, \ldots, t)}{B} = f(A/B, ..., \mathrm{t}) \qquad \text{(EQ 9)}$$

where *f( )* is a dimensionless function, and the .... denotes other variables like volatility, dividend yield, and so on, that are assumed to be independent of the values of *A* and *B*. This scaling or homogeneity means that *B* plays the role of a currency. As a consequence of the functional form of Equation 8, Euler’s theorem holds:

$$A\frac{\partial C}{\partial A} + B\frac{\partial C}{\partial B} = C \qquad \text{(EQ 10)}$$

or

$$C + A\Delta_A + B\Delta_B = 0 \qquad \text{(EQ 11)}$$

where $\Delta_A = -\frac{\partial}{\partial A} \cdot C$ and $\Delta_B = -\frac{\partial}{\partial B} \cdot C$ are the number of shares of A and B respectively needed to hedge a long position in the outpeformance option. In words, Equation 8 implies that the value of the outperformance option, the value of the *A-hedge* and the value of the *B-hedge* add to zero. This result is true even if the Black-Scholes model is invalid — for example, if stock evolution is non-lognormal. The main prerequisite for this to hold is that Equation 8 be valid: the outperformance call value must scale, so that its value in units of *B* is a function of *A/B* alone. If f*( )* contains other terms which depend on A or B alone, and not simply on their ratio, Equation 11 will no longer hold.

American-Style Options Valuation

For European-style options, Equation 6 above showed that

$$C_{\$}^{AB}(t) = C_B^{AB}(t)B_{\$}(t) = B_{\$}(t)BS(A_B, 1, d_B, \sigma(A_B), T-t)$$

As long as the scaling property of the previous section holds, this general decomposition is still valid, except that the *BS()* function must be replaced by the American option value *AM( )*:

$$C_{\$}^{AB}(t) = C_{B}^{AB}(t)B_{\$}(t) = B_{\$}(t)AM(A_B, 1, d_B, \sigma(A_B), T-t) \qquad \text{(EQ 12)}$$

You can use your favorite American-style valuation model (the binomial method, for example), to determine the value of *AM( )* as a call option with strike 1 and volatility $\sigma(A_B)$.

The Real World

Despite their elegance, outperformance options are relatively rare instruments. One reason is that the pricing volatility is determined by the correlation ρ_{AB} in Equation 4. Correlations are considered to be less stable than volatilities and are consequently harder to forecast. Market makers often take this into account by adding a larger risk premium to their theoretical prices for outperformance options than they do for standard options, which usually has the effect of making these instruments less attractive in practice than in theory.

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