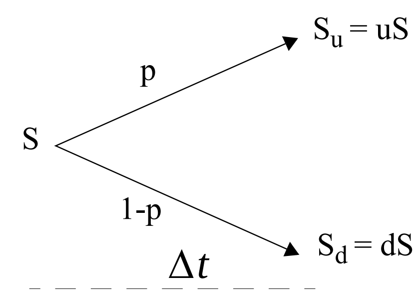
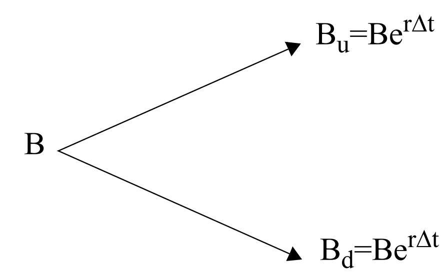
14.1 The Binomial Model for Options Valuation: One Period.Three Securities. p-Measure.

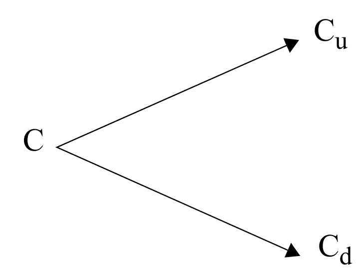
First the evolution for the stock: with "real" drift μ and vol σ. The parameter p is the **real-world probability** of things happening (assuming it exists), and μ is the actual drift. The probabilities used to calculate expectation values are the **p-measure**:



A riskless bond which evolves with no risk, hence:



A call option on the stock, with strike K and expiration T:



S and B span the final space of states, therefore a linear combination of S and B can **replicate any payoff** in the two final states, **no matter what their probabilities are.**

Analogy: If you know the price of a lottery ticket that pays off \$1 when a coin comes up heads, and you know the price of a lottery ticket that pays off \$1 when a coin comes up tails, you can figure out the fair price of a lottery ticket that pays off \$3.00 on heads and \$0.50 on tails, no matter whether the coin is fair or not, no matter what the probability of heads is.

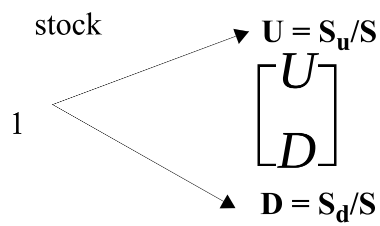
Switch to a New Basis of **State-Contingent Securities**

Use **vector notation** to represent securities $\vec{S}$ and $\vec{B}$ at the end of the period.

The symbols S and B without vector signs denote the values of the security.

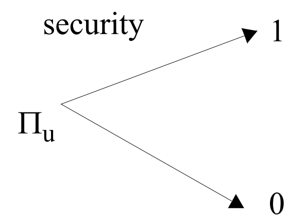
We can solve for $\overrightarrow{\Pi_u}$ and $\overrightarrow{\Pi_d}$ as portfolios or combinations of the stock $\vec{S}$ and the bond $\vec{B}$

$\$1$ invested in stock $\overrightarrow{1_S}$



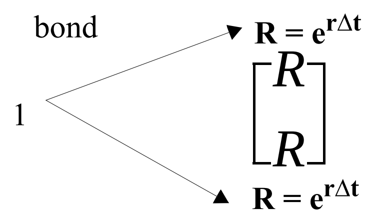
$$\begin{bmatrix} U \\ D \end{bmatrix}$$

$\overrightarrow{\Pi_u}$



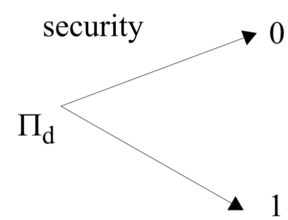
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\$1$ invested in bond $\overrightarrow{1_B}$



$$\begin{bmatrix} R \\ R \end{bmatrix}$$

$\overrightarrow{\Pi_d}$



$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note: $\overrightarrow{\Pi_u} + \overrightarrow{\Pi_d} \equiv \overrightarrow{(1,\ 1)}$ in future, same whether up or down happens $\equiv \dfrac{\overrightarrow{1_B}}{R}$

… New Basis

Write $\overrightarrow{\Pi_u} = \alpha \overrightarrow{1_S} + \beta \overrightarrow{1_B}$.

Then solve for α and β by looking at values in up and down state at the end of the period:

$$\begin{aligned} \alpha U + \beta R &= 1 \\ \alpha D + \beta R &= 0 \end{aligned} \quad \text{and so} \quad \begin{aligned} \alpha &= \frac{1}{(U-D)} \\ \beta &= \frac{-D}{R(U-D)} \end{aligned} \qquad \text{Eq.14.1}$$

Thus $\overrightarrow{\Pi_u} = \frac{1}{R}\left[\frac{R\bar{1}_S - D\bar{1}_B}{(U-D)}\right]$

We can similarly solve for $\overrightarrow{\Pi_d}$, to find $\overrightarrow{\Pi_u} = \frac{1}{R}\left[\frac{R\vec{1}_S - D\vec{1}_B}{(U-D)}\right]$ $\qquad \overrightarrow{\Pi_d} = \frac{1}{R}\left[\frac{U\overrightarrow{1_B} - R\overrightarrow{1_S}}{(U-D)}\right]$

$$\overrightarrow{\Pi_u} + \overrightarrow{\Pi_d} = \frac{1}{R}\overrightarrow{1_B} \text{ obviously}$$

… New Basis

The values of these state-contingent (Arrow-Debreu) securities **at the start of the period** are expectation values with a pseudo-probability q times the payoff of $1:

$$\Pi_u = \frac{1}{R}\left[\frac{R-D}{(U-D)}\right] \equiv \frac{\boldsymbol{q} \times 1}{\boldsymbol{R}} \qquad \Pi_d = \frac{1}{R}\left[\frac{U-R}{(U-D)}\right] \equiv \frac{(\mathbf{1}-\boldsymbol{q}) \times 1}{\boldsymbol{R}} \qquad \text{Eq.14.2}$$

where

$$q \equiv \frac{R-D}{U-D} \qquad 1-q \equiv \frac{U-R}{U-D} \qquad \text{Eq.14.3}$$

q is the **risk-neutral no-arbitrage "probability"** of going up, independent of the actual probability p.
Expectation values that use q use what is called the **q-measure**.

Note that

$$\overrightarrow{\Pi_{\mathrm{u}}} + \overrightarrow{\Pi_d} = \frac{\overrightarrow{1_B}}{R}$$

which means that a portfolio of the two together is a zero coupon bond that is worth $1 at the end of the period with no risk. The value of this portfolio at the start of the period is

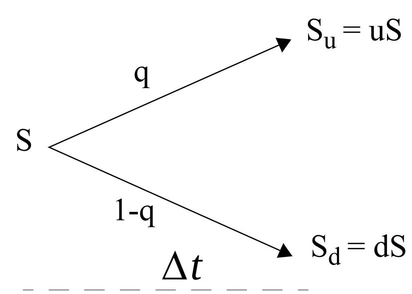
$$\Pi_u + \Pi_d = \frac{1}{R}$$

The Risk-Neutral *q*-Measure for Stock and its Relation to Forward Prices.

Note that the definition of $q \equiv \frac{R-D}{U-D}$ can be rewritten as $qU + (1-q)D = R$.

Multiplying both sides of the equation by the stock price *S*, we obtain.

$$S = \frac{qS_u + (1-q)S_d}{R} \qquad \text{Eq.14.4}$$

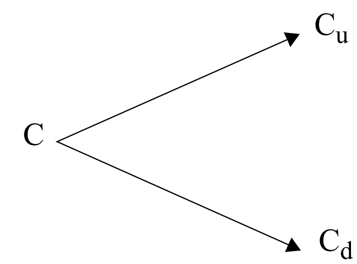


In the q-measure the current stock price is the risklessly discounted expected future value of the stock price, or, conversely, **the expected future stock price is the forward price**.

$$qS_u + (1-q)S_d = RS = F \qquad \text{Eq.14.5}$$

The Value of the Call Over One Binomial Period

Now any option C which pays a known amount C_u in the up-state and C_d in the down-state at the end of the period is replicated by



$\vec{C} = \overrightarrow{\Pi_u} C_u + \overrightarrow{\Pi_d} C_d$, which can be rewritten in terms of

$\vec{S}$ and $\vec{B}$

$$\vec{C} = (\overrightarrow{\pi_u} C_u + \overrightarrow{\pi_d} C_d) = \frac{R\overrightarrow{1_S} - D\overrightarrow{1_B}}{R(U-D)}(C_u) + \frac{U\overrightarrow{1_B} - R\overrightarrow{1_S}}{R(U-D)}(C_d) = \frac{C_u - C_d}{U-D}\overrightarrow{1_S} + \frac{UC_d - DC_u}{R(U-D)}\overrightarrow{1_B}$$

$$\vec{C} = \left(\Delta\vec{S} + B\overrightarrow{1_B}\right)$$ is the replicating portfolio for one period in terms of stock and bond

$$C = \Pi_u C_u + \Pi_d C_d \equiv \frac{q}{R} C_u + \frac{(1-q)}{R} C_d = \frac{qC_u + (1-q)C_d}{R}$$ is the value

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{\partial C}{\partial S}$$ hedge ratio // $B = C - \Delta S$ amount to be invested

$$S = \frac{qS_u + (1-q)S_d}{R}$$ definition of value of q

$$C = \frac{qC_u + (1-q)C_d}{R}$$ definition of value of C

All values of C and S are independent of p, depend only on q and the final states of the stock and option, **and are independent of the true probability p**.

These Formulas Mean:
Sharpe ratio of option = Sharpe ratio of stock!

We showed by replication: $\vec{C} = \Delta\vec{S} + B\overrightarrow{1_B}$ and $C = \Delta S + B$

Lets calculate the Sharpe ratios of the option and the stock

The expected final value of the call is a combination of expected final val of stock and bond

expected final value of C is $\Delta[S(1+\mu_S)] + [BR]$

But $B = C - \Delta S$ so

expected final value of C $\equiv (1+\mu_C)C = \Delta S(1+\mu_S) + (C - \Delta S)R$

$$(1+\mu_C) = \frac{\Delta S(1+\mu_S) + (C-\Delta S)R}{C} = \Delta\frac{S}{C}(1+\mu_S - R) + R$$

$$(1+\mu_C - R) = (1+\mu_S - R)\frac{S}{C}\Delta$$

But $\frac{S}{C}\Delta = \frac{S}{C}\left(\frac{C_u - C_d}{S_u - S_d}\right) = \left(\frac{C_u - C_d}{C}\right)\left(\frac{S}{S_u - S_d}\right) = \frac{\sigma_C}{\sigma_S}$

$$(1+\mu_C - R) = (1+\mu_S - R)\left(\frac{\sigma_C}{\sigma_S}\right)$$

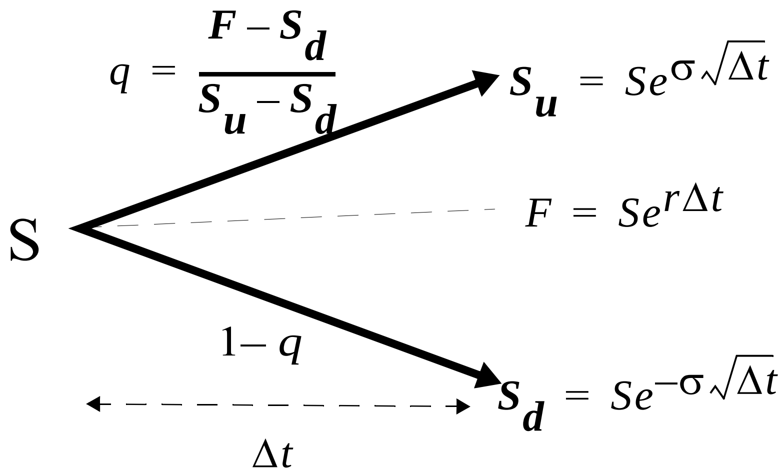
Note that R - 1 = r, the riskless rate

in other words .... $\frac{\mu_C - r}{\sigma_C} = \frac{\mu_S - r}{\sigma_S}$

This is how Black originally derived the BS equation

Summary: A Risk-Neutral Binomial Step in the q-Measure

The equations we use in the CRR binomial tree are



$$F = qS_u + (1 - q)S_d$$

$$C = \frac{qC_u + (1 - q)C_d}{R}$$

Summary: Replicating the Call Option Over One Binomial Period

Call Security: $\vec{C} = (\vec{\Pi}_u C_u + \vec{\Pi}_d C_d) = \Delta\vec{S} + B\overrightarrow{1_B}$

Values of Arrow-Debreu Securities: $\Pi_u = \dfrac{\boldsymbol{q}}{\boldsymbol{R}} \qquad \Pi_d = \dfrac{\boldsymbol{1-q}}{\boldsymbol{R}}$

Initial Value of Call: $C = \dfrac{qC_u + (1-q)C_d}{R}$

Thus you can replicate the call by owning Δ shares of stock and investing B dollars in a zero coupon bond that matures at the end of Δt, with

$$\Delta = \left(\frac{C_u - C_d}{S_u - S_d}\right)$$

$$B = \frac{S_u C_d - S_d C_u}{R(S_u - S_d)} = C - \Delta S$$

Δ is a calculus derivative in the limis   Eq.14.6

where all of these above parameters depend only on the final states of the stock and option, assumed known, **and are independent of the true probability *p***.

Call Value

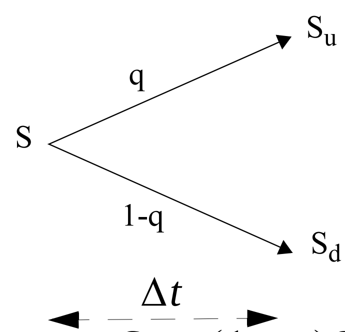
$$C = \frac{qC_u + (1-q)C_d}{R} \qquad \text{Eq.14.7}$$

$$S = \frac{qS_u + (1-q)S_d}{R} \qquad \text{Eq.14.8}$$

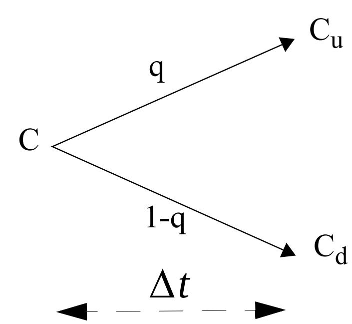
The value of the underlying stock and the replicated option are the discounted expected value of the terminal payoffs in the risk-neutral probability measure defined by q.

Equation 14.8 *defines* the risk-neutral measure q given the values of S, S_u and S_d.

Equation 14.7 *specifies* the value C in terms of the future option payoffs in measure q.



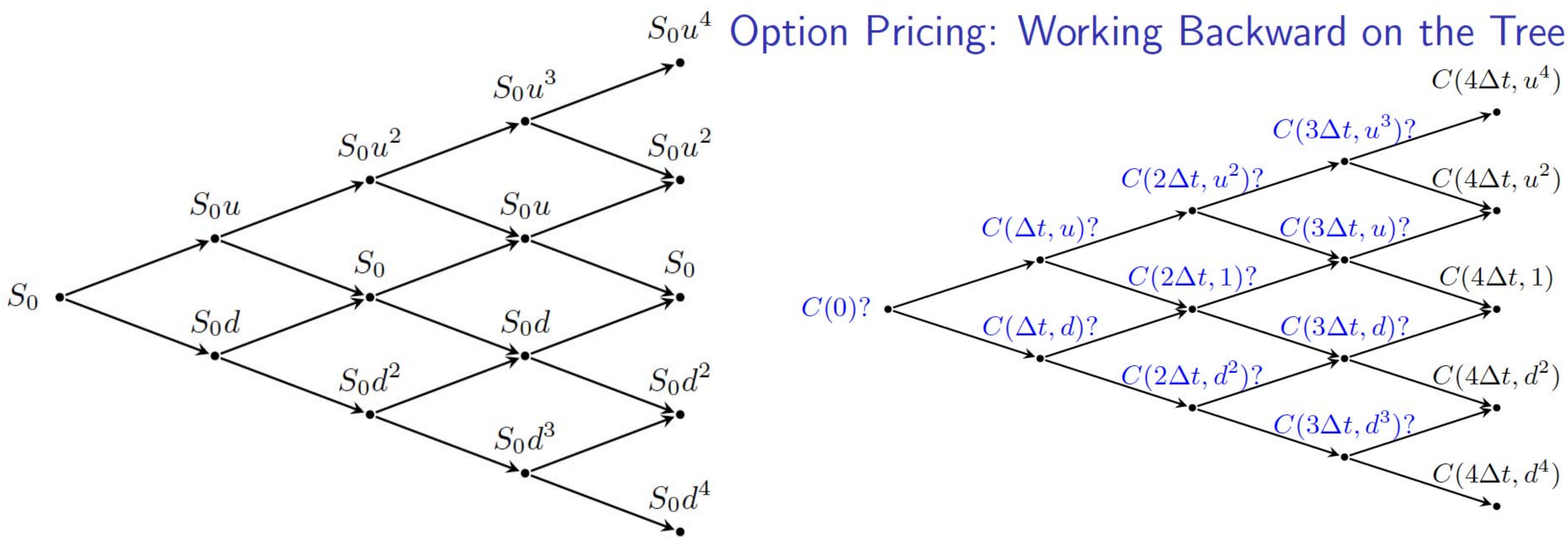
$$S = \frac{qS_u + (1-q)S_d}{R}$$



$$C = \frac{qC_u + (1-q)C_d}{R}$$

14.2 The Method on a Binomial Tree of Many Periods

Since we know the value of the European option at expiration, where $C_i(S_i) = [S_i - K]^+$, we can work our way backwards in time down the tree below and get the value of the option at the current time.



14.3 The Black-Scholes PDE

$$C(S, t, K, T, \sigma, r) \qquad \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

What does this equation mean? Rewrite it as $\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r\left(C - S\frac{\partial C}{\partial S}\right)$

Consider the RHS: $\left(C - S\frac{\partial C}{\partial S}\right)$ is the value of a hedged portfolio: long the call **C**, short $\Delta = \frac{\partial C}{\partial S}$ shares of stock **S**. Remember this removed all the stock risk from the binomial option method.

The change in the value of the hedged portfolio $\left(C - S\frac{\partial C}{\partial S}\right)$, from Ito, is obtained by letting C evolve through time and S evolve through time, but the number of shares $\frac{\partial C}{\partial S}$ remains fixed:

$$\frac{\partial C}{\partial t}dt + \cancel{\frac{\partial C}{\partial S}dS} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt - \cancel{dS\frac{\partial C}{\partial S}} = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right)dt$$

So the LHS of the equation is the actual return on the hedged portfolio.

The RHS of the equation is the riskless return on the hedged portfolio, $r\left(C - S\frac{\partial C}{\partial S}\right)$

Thus the Black-Scholes equation is actually stating:

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right) = r\left(C - \frac{\partial C}{\partial S}S\right)$$

Ito return on a delta-hedged portfolio = the riskless return on its initial value

If you hedge away all the risk you must earn the riskless return!

That's all the Black-Scholes equation says.

We will see that this equation is actually just the continuous-time equivalent of the discrete binomial equation

$$C = \frac{qC_u + (1-q)C_d}{R}$$

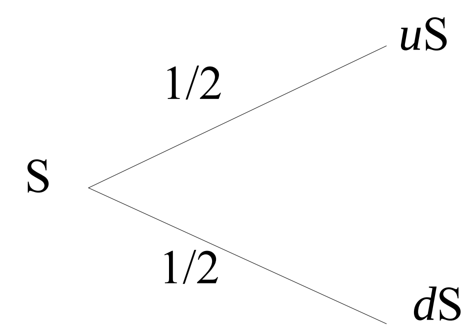
A backward equation!

which was equivalent to saying that the hedged portfolio earned the riskless rate.

BS PDE Derivation: Method 1: From Jarrow-Rudd Binomial Tree

The BS PDE can be obtained by taking the limit of the binomial pricing *q*-measure equation as $\Delta t \to 0$.

We'll use the JR choice of *u* & *d* and set $q = 1/2$ and $\mu = r - 0.5\sigma^2$ so that the stock price grows at the riskless rate *r*, as required for the q-measure. The log must grow at $\mu = r - 0.5\sigma^2$



$$u = e^{(r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} \qquad d = e^{(r - 0.5\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}$$

Since q = 1/2, option value is given by this **backward equation:**

$$e^{r\Delta t}C(S, t) = \frac{1}{2}C(S_u, t + \Delta t) + \frac{1}{2}C(S_d, t + \Delta t)$$

Expand in a Taylor series:

$$e^{r\Delta t}C = 0.5C\left(e^{(r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}}S, t + \Delta t\right) + 0.5C\left(e^{(r - 0.5\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}S, t + \Delta t\right)$$

Now $e^{(r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} \approx 1 + (r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t} + 0.5\sigma^2\Delta t \approx 1 + \sigma\sqrt{\Delta t} + r\Delta t$ so

$$e^{r\Delta t}C = 0.5C(S + S\sigma\sqrt{\Delta t} + rS\Delta t, t + \Delta t) + 0.5C(S - S\sigma\sqrt{\Delta t} + rS\Delta t, t + \Delta t)$$

$$(1 + r\Delta t)C = 0.5\left[C(S, t) + \frac{\partial C}{\partial S}S\{\sigma\sqrt{\Delta t} + r\Delta t\} + \frac{1}{2}\frac{\partial^2}{\partial S^2}C\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t\right] +$$

Thus to order Δt

$$0.5\left[C(S, t) + \frac{\partial C}{\partial S}S\{-\sigma\sqrt{\Delta t} + r\Delta t\} + \frac{1}{2}\frac{\partial^2}{\partial S^2}C\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t\right]$$

$$= C(S, t) + \frac{\partial C}{\partial S}S[r\Delta t] + \frac{1}{2}\frac{\partial^2}{\partial S^2}C\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t$$

and so

$$(\cancel{1} + r\Delta t)C = \cancel{C(S, t)} + \frac{\partial C}{\partial S}S[r\Delta t] + \frac{1}{2}\frac{\partial^2}{\partial S^2}C\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t$$

When you cancel the $C(S, t)$ term you obtain $Cr\Delta t = \frac{\partial C}{\partial S}\{rS\Delta t\} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t$ or

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} = rC$$

Black-Scholes Equation, depends only on r

It's just the continuous **backward equation** $C = \dfrac{qC_u + (1-q)C_d}{R}$ for the call price $C(S, t)$

The Black-Scholes PDE: Method 2: From Stochastic Calculus

The Merton method using Ito's Lemma and dynamic replication.

$$dS = \mu S dt + \sigma S dZ \qquad \text{in the } p\text{-measure}$$

Set up an instantaneously riskless portfolio, then require it earn the riskless rate by no-arbitrage:

$$\begin{aligned}
& C(S, t, K, T) && \text{option value} \\
\pi &= C(S, t, K, T) - \Delta S && \text{hedged portfolio long option short stock} \\
d\pi &= dC - \Delta dS && \text{growth in P\&L instantaneously} \\
&= \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2 - \Delta dS && \text{Ito} \\
&= \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S}(\mu S dt + \sigma S dZ) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt - \Delta(\mu S dt + \sigma S dZ) \\
&= \left[\frac{\partial C}{\partial t} + \left(\frac{\partial C}{\partial S} - \Delta\right)\mu S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right] dt + \left(\frac{\partial C}{\partial S} - \Delta\right)\sigma S dZ
\end{aligned}$$

Now hedge away all stock risk by choosing $\Delta = \frac{\partial C}{\partial S}$

Then if we hedge to remove stock risk,

$$d\pi = \left[\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right]dt$$

This is riskless: it is independent of dZ. Thus if we know the value of σ, then $d\pi$ replicates a riskless bond for an instant. So by the law of one price, it must earn the riskless rate:

$$d\pi \equiv \left[\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right]dt = r\pi dt \equiv (C - \Delta S)rdt = \left(C - \frac{\partial C}{\partial S}S\right)rdt$$

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt = \left(C - \frac{\partial C}{\partial S}S\right)rdt$$

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 = rC \qquad \text{Black-Scholes Equation} \qquad \text{Eq.14.9}$$

"Risk-neutral": *r* appears rather than μ

Risk-neutral means that all assets returns are expected to earn the riskless rate.

Actually, C could have been a stock, an option, a bond, any security dependent on S ….

Linear parabolic partial differential equation: $\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 = rC$

If we know the value on the boundaries, we can find the values at earlier times. (**Backward!**)

For a call, the boundary conditions are

at expiration: $C(S, T) = \max[S - K, 0]$

at low stock prices: $C(0, t) = 0$

at high stock prices: $C(S \to \infty, t) \to S - Ke^{-r(T-t)}$

We can solve this analytically or numerically.

Note: We've seen from equations above that this represents the binomial model taken in the limit as the time steps become small, and therefore the solution is the expected value of the payoff in the risk-neutral q measure in which the stock price grows on average at the riskless rate with volatility σ. (Feynman- Kac)