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Understanding Guaranteed Exchange-Rate Contracts in Foreign Stock Investments

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SUMMARY

U.S. investors who want a foreign stock or foreign index position may benefit by using guaranteed exchange-rate contracts. These options and forwards have a dollar payoff that does not depend upon the exchange rate prevailing at expiration. Contract holders are therefore not exposed to the exchange-rate risk of holding foreign stocks or indexes alone.

Nevertheless, the expected covariance between the exchange rate and the foreign asset's price has a subtle effect on the value of a guaranteed exchange-rate contract. A change in the expected covariance will cause a change in the contract's value.

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INTRODUCTION

Suppose you expect a certain West German stock to appreciate significantly over the next year. If you're right, and you're a U.S. dollar investor, how can you be sure to capture this appreciation for your portfolio?

Buying the stock won't work, nor will taking a long position in it through a forward contract or a call option. In order to purchase the stock in the German market you have to pay for it by buying marks at today's exchange rate. One year hence, your rate of return *in marks* depends upon the price at which you sell the stock in the German market. But your return *in U.S. dollars* depends upon the rate at which you can convert the marks you receive back to dollars. So you have been investing in the German mark in dollar terms as well as in the stock.

To avoid having your return depend on the mark's performance, you need a guarantee that you can close out your stock position at an exchange rate close to the one that prevailed when you opened your position. Then you can convert your stock position's profit or loss to dollars without worrying about the prevailing dollar-mark exchange rate.

This paper explains how to use, value, and think about *guaranteed exchange-rate forward and option contracts* in foreign stock or index investments.

DEFINING THE CONTRACTS

An ordinary foreign stock forward contract is an agreement to buy a stock on a certain date at a certain delivery price in a specified currency. On the delivery date the contract's value is the difference between the stock's price and the delivery price in the foreign currency, converted to dollars at the *prevailing* spot exchange rate.

In the same way, the value on the delivery date of a *guaranteed exchange-rate forward contract* on a stock is the difference between the stock's price and the delivery price in the foreign currency, but the difference is converted to dollars at a *predetermined* exchange rate.

More formally, a guaranteed exchange-rate (GER) forward contract on a foreign stock is an agreement to receive on a certain date the stock's prevailing price in exchange for a predetermined foreign-currency delivery price, with both prices converted to dollars at a predetermined exchange rate.

Similarly, a GER call (put) option on a foreign stock gives you the right to receive (deliver) by a certain date the stock's price in exchange for a predetermined foreign-currency delivery price, with all prices converted to dollars at a predetermined exchange rate.

Payoffs for GER forward and option contracts are independent of the foreign exchange rate. Therefore the contracts allow you to capture a foreign stock's return in your own currency.

The theoretical value of a GER contract is independent of the actual value of the exchange rate at any instant during its life. But surprisingly, the contract's value *is* influenced – albeit subtly – by the expected covariance between the exchange rate and the stock price. We explain this below.

VALUING A FORWARD CONTRACT

The Method

The GER forward is a derivative security whose dollar value depends upon the foreign-currency price of the underlying stock. As with other derivatives, the traditional way to think about the contract's value is to look at how to hedge its *dollar value* against small moves in the price of the underlying stock. Let's look at this step-by-step.

You can hedge the forward against movements in the stock's price by shorting an appropriate number of shares of the stock. But this introduces exchange-rate risk since the dollar value of the short position depends on the exchange rate. To neutralize this risk, you keep the cash proceeds of the short sale invested in the foreign currency; a move in the exchange rate alone then has an equal and opposite effect on the dollar value of the short stock and long foreign currency positions. In this way, you can set up a three-part portfolio (the GER forward, the short stock position, and the foreign currency) whose dollar value will not change if either the foreign stock price or the foreign exchange rate changes in the next instant by a small amount.

This portfolio is therefore momentarily riskless. Assuming that the market does not allow riskless arbitrage, the portfolio must instantaneously earn the U.S. riskless rate. Consequently, you can value the portfolio by discounting its expected payoff *as if* you lived in a world where all investments, whatever their risk, were expected to earn this riskless rate.[1]

Because the GER forward is part of the portfolio, you can also calculate its value as though you lived in that world. You can pretend to live in it from a mathematical point of view by adjusting the centers of the probability distributions of your risky securities at future times so that their expected returns are riskless. Then it is a straightforward matter to value the con-

[1] F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May/June 1973), pp. 637-654. See also, for example, John Hull, *Options, Futures and Other Derivative Securities* (Prentice-Hall, 1989), pp 95-99.

tracts by discounting their expected payoffs.

Note that we are using hedging to value the contract. We hedged the GER contract against stock price moves by taking a short position in the foreign stock. We then hedged the sensitivity of the dollar value of this foreign stock to exchange-rate moves by taking a long position in the foreign currency. Therefore, the contract's value can depend upon the *properties of the probability distribution* of the future values of the exchange rate, even though it is insensitive to the *current value* of the exchange rate.

Let's apply this method to a GER forward on a German stock. The following table defines our notation.

t	– time to delivery
$S(0)$	– stock price in German marks today
$S(t)$	– stock price in marks at delivery
D	– continuous dividend rate of the stock
$X(t)$	– spot dollar value of the mark at delivery
K	– stock's delivery price in marks
X_0	– value of the mark in dollars used to convert the GER payoffs to dollars
$F(0)$	– dollar value of the GER forward today
$S_F(t)$	– stock's GER forward price in marks
$r_\$$	– U.S. riskless interest rate
r_g	– German riskless interest rate
σ_s	– estimated return volatility of the stock's price in marks
σ_x	– estimated return volatility of the mark's value in dollars
σ_{xs}	– estimated covariance between returns of the mark in dollars and the stock price in marks
ρ_{xs}	– estimated correlation coefficient $\sigma_{xs}/(\sigma_x\sigma_s)$

At delivery t years from now, the dollar value of the forward contract's payoff is

$$(S(t) - K)X_0 = S(t)X_0 - KX_0 \qquad (1)$$

Notice that the payoff is converted to dollars at the guaranteed rate.

The value of the GER forward contract today, $F(0)$, is the discounted expected value of the contract's dollar payoffs in equation (1) in the simplified world where all investments are expected to earn the U.S. riskless rate $r_\$$. To calculate the expected value of equation (1), we must center the future probability distributions of the dollar value of the mark and the dollar value of the German stock so that the expected returns on these investments are $r_\$$.

Centering the Probability Distribution

We begin by assuming that the mark and the stock have lognormally distributed probability distributions at future times in the simplified world with riskless expected returns. Let the dollar value of the mark at delivery, $X(t)$, be lognormally distributed with a volatility σ_x and a mean growth rate r_x. Similarly, let the stock price in marks, $S(t)$, be lognormally distributed with volatility σ_s and a mean growth rate r_s. Initially we don't know what values to assign to the two growth rates r_x and r_s in the simplified world. But we do know that our dollar-based investments in two assets, the mark and the stock, must have mean growth rates equal to the riskless rate $r_\$$ in this world. We can use these two constraints to pin down the values of r_x and r_s.

First let's fix r_x by calculating the expected growth rate of an investment in the German mark. An investment of $X(0)$ dollars gets you one mark today which you can invest at the German riskless rate r_g. In addition, the value of a mark in this world is expected to grow at a rate r_x. Your total dollar investment therefore grows at an expected rate $r_x + r_g$. You now need to center the mark's probability distribution by choosing

r_x so that this total growth rate is $r_\$$. In other words, choose r_x so that

$$r_\$ = r_x + r_g \tag{2}$$

Therefore, you must choose the center of the probability distribution for $X(t)$ so that the growth rate r_x is

$$r_x = r_\$ - r_g \tag{3}$$

Now let's fix r_s in this world by calculating the expected growth rate of an investment in the German stock. Suppose you buy one share of the stock today by paying $S(0)X(0)$ dollars for $S(0)$ marks worth of stock converted to dollars at today's exchange rate $X(0)$. If the stock pays dividends at a continuous rate D, you can reinvest them and own e^{Dt} shares of stock at delivery, each worth $S(t)$ marks. The dollar value of your position is $e^{Dt}S(t)X(t)$. Notice that it depends upon the product of two lognormally distributed variables $S(t)$ and $X(t)$. Its expected value is

$$E[e^{Dt}S(t)X(t)] = e^{(D+r_s+r_x+\sigma_{xs})t}S(0)X(0) \tag{4}$$

where σ_{xs} is the expected covariance between the returns on the stock and the mark. σ_{xs} affects the expected value because the mean of the product of two probability distributions differs from the product of their means. As you did with the mark's distribution, you now center the S(t) probability distribution by choosing r_s so that the value of your initial investment in the stock is expected to grow at the riskless rate $r_\$$ to the value $e^{r_\$ t}S(0)X(0)$. This means that the exponent in equation (4) must satisfy

$$r_\$ = D + r_s + r_x + \sigma_{xs} \tag{5}$$

You can satisfy this by centering the $S(t)$ distribution so that

$$r_s = r_\$ - r_x - D - \sigma_{xs} \tag{6}$$

You can substitute for r_x from equation (3) to show that

$$r_s = r_g - D - \sigma_{xs} \tag{7}$$

In the simplified world where we calculate option prices by discounting expected payoffs, equation (7) is the growth rate for the stock value $S(t)$ in marks.

Finding the Value

Let's summarize our results. You can find the fair dollar value of the GER forward or option by discounting the expected value of its dollar-valued payoffs at the U.S. riskless rate. However, you must center the lognormal probability distributions of the mark and the stock so that the expected return on any investment is equal to the U.S. riskless rate.

The required lognormal distribution for the mark has volatility σ_x and mean

$$E[X(t)] = X(0)e^{(r_\$ - r_g)t} \tag{8}$$

The required distribution for the stock has volatility σ_s and mean

$$\begin{aligned} E[S(t)] &= S(0)e^{(r_g - D - \sigma_{xs})t} && (9) \\ &= S(0)e^{(r_g - D')t} && (10) \end{aligned}$$

where

$$D' = D + \sigma_{xs} \tag{11}$$

The stock price in marks is expected to grow at a rate that is reduced by D', the sum of the real dividend rate and the covariance between the returns on the mark and the stock. In this world, it's as though the stock pays out additional dividends at a rate σ_{xs}. Note, for future use, that $X_0S(t)$ has a lognormal distribution with the same volatility as $S(t)$ and the same additional dividend rate.

Now that you know the stock's probability distribution at delivery, you can calculate today's discounted expected value of the GER forward's payout in equation (1) by using equation (10) to obtain the value

$$
\begin{aligned}
F(0) &= E\left[S(t)X_0 - KX_0\right]e^{-r_s t} && (12)\\
&= \left[S(0)X_0 e^{(r_g - D')t} - KX_0\right]e^{-r_s t} && (13)
\end{aligned}
$$

We define the forward price $S_F(t)$ for delivery at time t as the value in marks of the GER delivery price K that makes the forward contract's value $F(0)$ equal to zero:

$$S_F(t) = S(0)e^{(r_g - D')t} \qquad (14)$$

By comparing equations (10) and (14), you can see that the forward price is simply the expected value of the stock in the simplified world.

Remarks

The GER forward price in equation (14) is almost the same as that of an ordinary mark-denominated forward contract whose forward price is $S(0)e^{(r_g - D)t}$ marks. The key difference is that the GER forward price uses the effective dividend rate D' rather than the true dividend rate D. Thus, it depends upon σ_{xs} (the expected covariance of the mark's return with the stock's return in marks), even though it does not depend upon the value of the mark at any particular time during the contract's life.

Note that either an increase in r_g or a decrease in σ_{xs} causes the forward price in (14) to increase.

When the covariance is positive, D' is greater than D. In this case the forward price of the GER forward is lower than that of an ordinary (non-GER) forward contract with forward price $S(0)e^{(r_g - D)t}$ in the German market. It's as though the stock were simply paying a higher average dividend rate $D' = D + \sigma_{xs}$.

The reduced forward price for a positive covariance makes sense intuitively if you think about the following argument. A positive covariance means that, on average, the mark is expected to strengthen relative to the dollar when the stock price in marks rises. Because of this, an ordinary forward whose delivery price is specified in marks and whose payout is converted to dollars at the spot exchange rate at delivery has a greater expected dollar value than a guaranteed exchange-rate forward. So you should expect a lower forward delivery price if the exchange rate is guaranteed.

VALUING AN OPTION

You can now easily value a GER put or call with a strike price of K German marks. The payoff of the put with European-style exercise is

$$max\,[X_0K - X_0S(t), 0] \tag{15}$$

This is the same as the payoff of a put struck at KX_0 dollars on an imaginary stock whose price distribution is that of $X_0S(t)$ with volatility σ_s and mean growth rate $r_g - D'$. The dollar value of the put P_0 is the present value of the expected payoff in equation (15):

$$P_0 = E\,[\,max[X_0K - X_0S(t), 0]\,]\,e^{-r_{\$}t} \tag{16}$$

Its value is

$$P_0 = \left[KN(-d_2) - S(0)e^{(r_g - D')t}N(-d_1)\right]X_0e^{-r_{\$}t} \tag{17}$$

where the first term represents the probability of the put being in the money, and the second is the expected value of the stock when the put is in the money. Here $N()$ is the cumulative lognormal distribution and

$$d_1 = \frac{ln(S(0)/K) + (r_g - D' + \sigma_s^2/2)t}{\sigma_s\sqrt{t}} \tag{18}$$

$$d_2 = d_1 - \sigma_s\sqrt{t} \tag{19}$$

It's easy to verify that a portfolio consisting of a long call and a short put, both struck at K marks with guaranteed exchange rate X_0, is equivalent to a GER forward and has the same value as the forward in equation (13).

American-style GER options can be valued on a binomial tree by using the standard Cox-Ross-Rubinstein method applied to a stock with price distribution given by $X_0S(t)$, strike X_0K dollars, and dividend rate D', and then considering the possibility of early exercise at each binomial node.

AN EXAMPLE

To get a feel for the prices of GER options, and to compare them with the prices of plain foreign stock options, walk through the following example with us.

Suppose that today we write two one-year European options on a German stock: (1) a plain option whose strike and payoff are in marks; and (2) a GER option with payoff in dollars. Both options are struck at the current stock price of S marks per share. The guaranteed exchange rate X_0 is equal to the current exchange rate X.

We assume that: (1) the stock pays no dividends; (2) the annual volatility of the stock's return in marks is 20%; (3) the annual volatility of the dollar/mark exchange rate's return is 10%; and (4) the U.S. interest rate is 9% per year, continuously compounded.

We show below put and call prices, as a percentage of the current stock price, for both a plain option and a GER option. We look at three continuously compounded German interest rates: 7%, 9%, and 11% per year, and at five values of the correlation coefficient ρ_{xs}.

Guaranteed Exchange-Rate European Option Values as a Percentage of the Underlying Stock's Price

Correlation Coefficient	$r_\$ = 9\%$ $r_g = 7\%$ Put	Call	$r_\$ = 9\%$ $r_g = 9\%$ Put	Call	$r_\$ = 9\%$ $r_g = 11\%$ Put	Call
1.0	5.35	10.04	4.69	11.31	4.08	12.68
0.5	5.01	10.66	4.37	11.99	3.79	13.40
0.0	4.69	11.31	4.08	12.68	3.52	14.15
−0.5	4.37	11.99	3.79	13.40	3.26	14.92
−1.0	4.08	12.68	3.52	14.15	3.02	15.71
Plain Option	4.78	11.54	4.08	12.68	3.45	13.87

As we noted in the "Remarks" section on page 8, an increase in German interest rates or a decrease in the correlation coefficient, all else equal, causes the GER forward price $S_F(0)$ in equation (14) to increase. As a result, the call is more likely to be in-the-money at expiration, and its price increases. Similarly, the put price decreases.

APPENDIX: AN ALTERNATIVE APPROACH

Introduction

Another way to explain the pricing of a guaranteed exchange-rate option is to use a hedging argument. With this argument we show that a GER option on one share of a German stock is equivalent to an option on one share of a three-part synthetic dollar security.

The security's price is equal to the price of a German stock in marks converted to dollars at the guaranteed exchange rate. Its price return volatility is equal to the mark price return volatility of the German stock. But its dividend payments differ from those of a German stock. We show how.

One share of this synthetic security is a portfolio consisting of positions in: (1) shares of a German stock; (2) dollars; and (3) marks. We find the exact amount of each of these three components. Once we know them, we find the amount of dividends the synthetic security pays.

In the end we have a simple recipe for pricing and hedging GER options: Using your plain stock option calculator, you find the price and delta of a plain option on the synthetic dollar security struck at the GER option strike converted to dollars at the guaranteed exchange rate. The price is your GER option price. To hedge the GER option, you short "delta" shares of the synthetic security.

Hedging a GER Option

Let A be a German stock whose price in marks at time t is $S(t)$. Assume it pays no dividends.

Consider a GER option on stock A struck at K marks per share with guaranteed exchange rate X_0 dollars per mark. Write O for the dollar price of this option. This price is independent of the current exchange rate. How can we hedge this option?

Write O_Δ for the option's dollar delta due to the change in A's price in marks. When S changes by the small amount ΔS marks, the option price changes by $O_\Delta \, \Delta S$ dollars.

To hedge the GER option's dollar delta we short:

$$\frac{1}{X} O_\Delta$$

shares of **A**.

To hedge the exchange rate risk of the short position in stock **A** we add marks to our hedge portfolio in the amount:

$$\frac{1}{X} O_\Delta \, S$$

Creating the Synthetic Security

Now we construct a dollar security $\mathbf{A}_{syn}$. We want to design it so that an option on one share of $\mathbf{A}_{syn}$ struck at $X_0 K$ dollars per share has the same value as a GER option on one share of **A** struck at K marks per share with guaranteed exchange rate X_0.

For this to be true these two options must have the same intrinsic value at any point in time before expiration, for any future price behavior of **A**. Therefore the price of $\mathbf{A}_{syn}$ must always be equal to the price of **A** in marks converted to dollars at the fixed exchange rate X_0.

Write $S_{syn}(t)$ for the dollar price of $\mathbf{A}_{syn}$ at time t. It is given by:

$$S_{syn}(t) = X_0 \, S(t)$$

Write $O_{\Delta syn}$ for the $\mathbf{A}_{syn}$ option's dollar delta due to the change in the price of $\mathbf{A}_{syn}$. It is related to O_Δ:

$$O_\Delta = X_0 \, O_{\Delta \, syn}$$

To hedge an $\mathbf{A}_{syn}$ option, or its equivalent GER option on **A**, we need to short $O_{\Delta syn}$ shares of $\mathbf{A}_{syn}$. We can obtain the same result by shorting $O_{\Delta syn}$ shares of a fund where one share represents a portfolio of:

$$(1)\quad \frac{X_0}{X}\ \mathbf{A} \text{ shares, and } (2)\quad \frac{X_0}{X}\, S \text{ marks borrowed.}$$

In either case we must short the same number of shares. If we hedge by shorting $\mathbf{A}_{syn}$ shares, the price is S_{syn}. If we hedge by shorting fund shares, the shares are worthless.

We can create a riskless portfolio worth S_{syn} dollars by combining one share of $\mathbf{A}_{syn}$ with a short position in one share of the fund described above. This portfolio is equivalent to a position in S_{syn} dollars. Likewise, one share of $\mathbf{A}_{syn}$ is equivalent to a portfolio composed of one share of the fund plus a position in S_{syn} dollars.

Thus we can create one share of $\mathbf{A}_{syn}$ by:

- investing $X_0\, S(t)$ dollars at the U.S. interest rate;
- borrowing $X_0/X(t)\, S(t)$ marks at the German interest rate; and
- using the borrowed marks to buy $X_0/X(t)$ shares of stock $\mathbf{A}$.

We must adjust these three components as the price of $\mathbf{A}$ and the $/DM exchange rate change, and as we pay and receive interest. At any point in time, the portfolio is worth $X_0\, S(t)$ dollars.

Now that we've shown how to construct $\mathbf{A}_{syn}$ we can view a GER option on $\mathbf{A}$ as a plain option on $\mathbf{A}_{syn}$. To price this plain option we need the price return volatility of $\mathbf{A}_{syn}$, which, by construction, is equal to the mark price return volatility of $\mathbf{A}$. We also need to find out about $\mathbf{A}_{syn}$'s dividend payments.

The Synthetic Security's Dividends

To find $\mathbf{A}_{syn}$'s dividend payments over a short period of time, we begin by finding its total return. This total return has two components: gain due to change in price and dividend payments.

Let's find $\mathbf{A}_{syn}$'s total return over a short time period Δt as the price of $\mathbf{A}$ changes by ΔS, as the exchange rate changes by ΔX, and as we pay and receive interest. This total return is the sum of total returns on each of $\mathbf{A}_{syn}$'s three components.

The net dollar amount of the interest rate payment received over time Δt is:

$$X_0 S(t)(r_{\$} - r_g)\Delta t$$

If the exchange rate changes by ΔX, the profit on the borrowed marks is:

$$-\frac{X_0}{X} S(t)\Delta X$$

The total gain on the position in stock $\mathbf{A}$ is due to the change in $\mathbf{A}$'s dollar price. This change $\Delta(XS)$ has three parts:

$$\Delta(XS) = X\,\Delta S + S\,\Delta X + \Delta X\,\Delta S$$

The first part is due to $\mathbf{A}$'s price change assuming a fixed exchange rate. The second part is due to a change in the exchange rate assuming a fixed mark price for $\mathbf{A}$. The third part is a cross-over term.

We find $\mathbf{A}_{syn}$'s total return by combining gains on each of its three components:

$$X_0 \Delta S + X_0 S \left\{ r_{\$} - r_g + \frac{1}{\Delta t}\frac{\Delta X}{X}\frac{\Delta S}{S} \right\} \Delta t$$

The first part of this total return, $X_0\,\Delta S$, is due to the change in $\mathbf{A}_{syn}$'s share price. The second part is the amount of dividends paid.

Thus, over a short time period Δt, one $\mathbf{A}_{syn}$ share pays a dividend whose dollar amount is:

$$X_0 S\left\{r_\$ - r_g + \frac{1}{\Delta t}\frac{\Delta X}{X}\frac{\Delta S}{S}\right\}\Delta t$$

This dividend is the net amount of dollars taken from one $\mathbf{A}_{syn}$ share in the process of adjusting the amount of each of its three components.

Assume that the return of the mark's value in dollars and of $\mathbf{A}$'s price in marks have a joint normal probability distribution and are correlated.

The annualized volatilities of returns are σ_x for the mark's dollar price and σ_s for $\mathbf{A}$'s price. The correlation coefficient of these two returns is ρ_{sx}.

Given these assumptions, as our time interval Δt approaches zero, we obtain:[2]

$$\frac{1}{\Delta t}\frac{\Delta X}{X}\frac{\Delta S}{S} = \sigma_s\sigma_x\rho_{sx}$$

Finally, we find that $\mathbf{A}_{syn}$ pays a continuous dividend at a fixed rate D_{syn}:

$$D_{syn} = r_\$ - r_g + \sigma_s\sigma_x\rho_{sx}$$

Depending on the spread between U.S. and German interest rates, and on the correlation term, this dividend rate can be positive or negative.

[2]For a more detailed explanation, see Hull, p. 176.

It's easy to extend our analysis to a dividend-paying German stock. For a stock **A** paying a continuous dividend at a fixed rate D, the corresponding synthetic dollar security $\mathbf{A}_{syn}$ pays dividends at a rate:

$$D_{syn} = D + r_{\$} - r_g + \sigma_s \sigma_x \rho_{sx}$$

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