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Investing in Volatility

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SUMMARY

Index futures contracts provide pure index exposure. Index options provide exposure to both the index and its volatility. What if you simply want volatility exposure?

The theoretical analysis of index volatility is evolving along a path similar to that taken by interest rate modeling in the fixed income options world more than a decade ago. This article outlines the concepts driving this evolution, whose natural endpoint will be the ability to market, value, trade and hedge pure index volatility, uncontaminated by exposure to the level of the index itself.

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INTRODUCTION

A Brief History of Interest Rates

In the beginning was the bond, its value measured simply by its price. Soon, analysts invented better measures of relative bond value: *current yield*, which led to *yield to maturity*, followed by the *term structure of yields*, the *zero coupon yield curve*, and finally *forward rates* and the *forward rate curve*. Their importance reflected the fact that these future rates can be locked in, once and for all, using bond portfolios today. From then on, every interest rate trader and analyst carried in his head an abstract forward rate curve.

Thinking in terms of forward rates stimulated the development of new derivative instruments which mirrored the underlying reality of forward rates; Eurodollar futures contracts and interest rate swaps are just two examples. But as far as the valuation and analysis of these instruments was concerned, the forward rate curve was a deterministic, static *parameter* whose future evolution had no impact on the current instrument value.

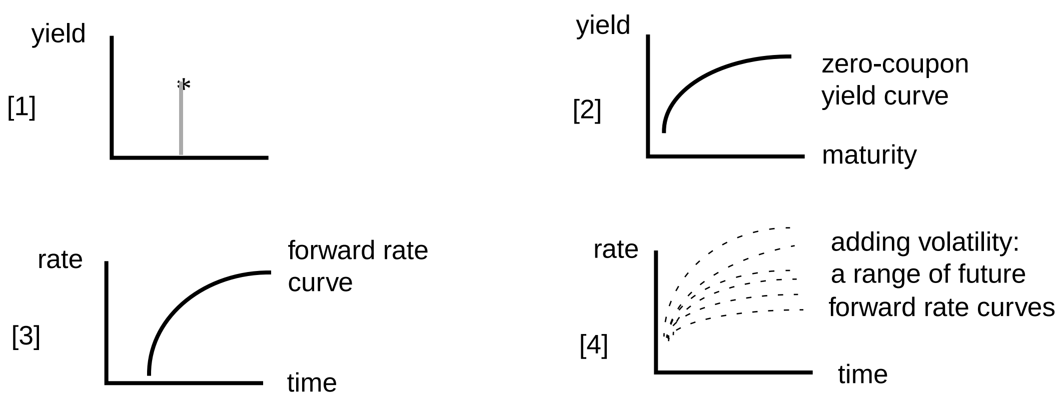
The history of interest rate analysis in the 1980's and 1990's has been the story of increasingly successful attempts by modelers to breathe realistic life and movement (that is, volatility) into the evolution of the forward rate curve, without violating theoretical arbitrage bounds. As traders' confidence in these valuation models and their hedges grew, these efforts led to more liquid and better tailored varieties of interest rate caps, swaptions, range notes and related interest rate derivatives whose values depend not only on the yield curve today, but on its volatility as well. Figure 1 contains a schematic history of interest rate modeling.

The Past and Future of Index Volatility

There is a fundamental similarity between the role of interest rates in the pricing of bonds and the role of volatility in the pricing of index options. The term structure of interest rates provides the parameters that determine bond values; the more complex term and strike structure of implied volatilities provide the parameters that determine options values.

Volatility first entered the options world as a single parameter in the Black-Scholes formula. As each bond was characterized by its own yield to maturity, so each option had its own implied volatility. Soon, market participants began to abstract a term and strike structure of implied volatility – an *implied volatility surface* for each option strike and expiration – that was the two-dimensional analog of the *yield curve* for bonds. As market yield curves imply a curve of attainable forward rates which can be theoretically locked in via bond portfolios, so

FIGURE 1. The evolution of interest rate modeling.



market implied volatilities uniquely determine a surface of attainable future *local volatilities*, varying with market level and future time[1], which can be theoretically locked in by index options portfolios.

The final step in this evolution, still to come, is to add realistic volatility to the local volatility surface itself, again without violating any arbitrage constraints. We expect a better understanding of the volatility of volatilities to lead to the ability to produce, market, trade and hedge a wide array of volatility derivatives, and so make possible the investment in pure volatility. Figure 2 contains an illustration of the evolution of volatility models.

WHAT WE MEAN WHEN WE TALK ABOUT VOLATILITY

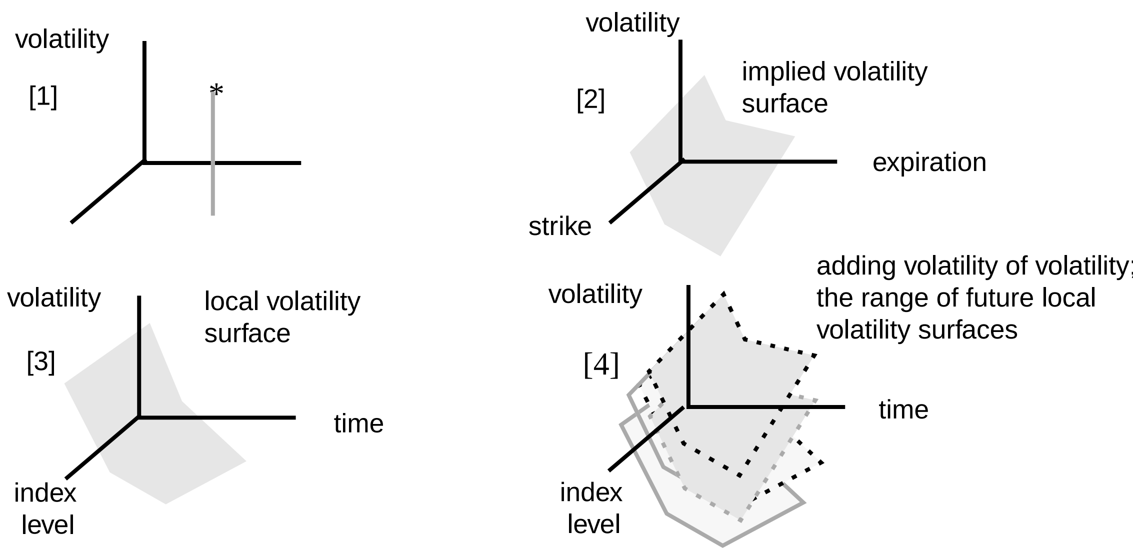
People use the world “volatility” to mean several related but distinct concepts. From now on, for clarity, we shall always qualify volatility with the adjective “realized,” “implied,” or “local.”

The *realized volatility V* of an index over some period is the annualized standard deviation of its daily returns over that period.

The *implied volatility* $\Sigma(K,T)$ of an index, as implied by the current price of a particular European-style option with strike K and expiration T, is the volatility parameter that, when entered into the Black-Scholes formula, equates the model value to the option price.

1. See Derman, Kani and Zou [1996].

FIGURE 2. The evolution of index volatility modeling.



The *local volatility* $\sigma(S,T)$ of an index at some future market level S and time T is the future volatility the index must have at that time and market level in order to make current options prices fair. Local volatilities can be extracted from the set of all implied volatilities $\Sigma(K,T)$ at a given time, and locked in by trading portfolios of currently available options.

The analogy with interest rates is helpful in understanding these distinctions.

> Realized interest rates are the actual interest rates that come to pass during some period; they are the analog of realized volatility.
>
> The yield to maturity of the bond is its implied yield; as implied volatility translates into an option price through the Black-Scholes equation, so yield to maturity translates into a bond price through the present value formula.
>
> Finally, forward rates are the future rates that must come to pass to justify current yields to maturity; they are the future rates that can be locked in by trading current bond portfolios. Local (sometimes called “forward”) volatilities are their volatility analog.

INVESTING IN INDEX VOLATILITY

You can trade volatility in all its forms: implied, local and realized. In brief, implied volatility is the price you must pay today to get exposure to future realized volatility. Local volatility is the price you can lock in today for future delivery of exposure to future realized volatility. We elaborate on these principles below.

Trading Implied Volatility With Options

Implied volatility is the market price, denominated in Black-Scholes currency, that you pay today to gain exposure to index volatility over some period starting today. Implied volatilities for most global equity indexes vary quite strongly with strike and expiration, displaying the well-known volatility smile. But the exposure to volatility you obtain by owning index options is impure: the options provides index level risk, dividend risk, interest rate risk *and* volatility risk. The difficulty lies in removing the exposure to everything except the volatility.

Isolating Local Volatility with "Gadgets"

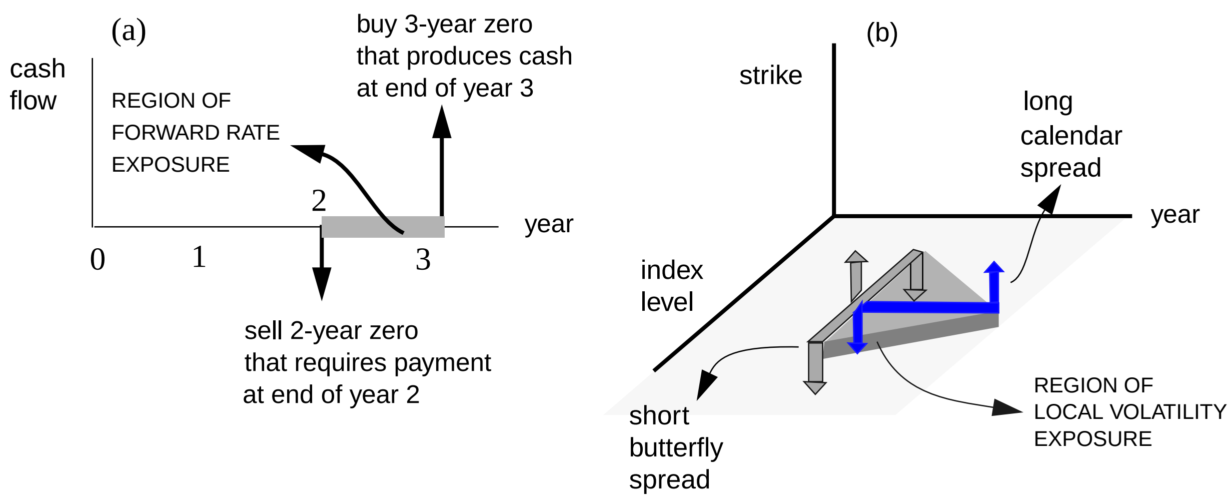
Local or forward volatility is the market price of volatility you can lock in today to obtain volatility exposure over some specific range of future times and market levels.

As usual, the interest rate analogy helps. In the interest rate world, for example, you can lock in future interest rates between two years and three years by going simultaneously long a three-year zero-coupon bond and short a two-year zero-coupon bond, such that the present value of the total bond portfolio is zero. In this way you lock in the one-year forward rate two years from now. This bond portfolio has exposure only to this particular forward rate. As time passes, this portfolio increases in value as the forward rate drops, and decreases as it rises. You can think of this portfolio, which we call an *interest rate gadget*, as a synthetic Eurodollar interest rate futures contract that can be used to hedge exposure to forward rates. Figure 3a illustrates the interest rate gadget.

Something similar can be done with options[2]. No listed local volatility futures are yet available. But, we can define a portfolio of standard options, called a *volatility gadget*, that serves as the volatility analog of a Eurodollar futures contract[3]. A volatility gadget is a small portfolio of

2. See Kani, Derman and Kamal [1996].
3. In the interest of clarity, we allow ourselves the luxury of a somewhat cavalier description of the gadget, which is only strictly correct if interest rates are zero and the strikes in the option spreads are infinitesimally close. For a more general (and more complex) treatment, see the reference in Footnote 2.

FIGURE 3. (a) A schematic depiction of an interest rate gadget that provides exposure to forward rates between years 2 and 3. (b). A schematic depiction of a volatility gadget that provides exposure to local volatility in the shaded index region. Up (down) arrows correspond to long (short) positions.



standard index options, created out of a long position in a calendar spread and a short position in an overlapping butterfly spread that has zero total options value. This gadget is sensitive to (forward) local index volatility only in the region between the strikes and expirations of the spreads in the gadget. Figure 3b illustrates the volatility gadget. By buying or selling suitably constructed gadgets corresponding to different future times and index levels, we can theoretically hedge an index option portfolio against changes in future local volatility.

Trading Realized Volatility By Hedging Options

Once you own an option, you have “paid implied volatility” to obtain exposure to both volatility and index level, ignoring for now the smaller risks due to variation in dividend payments and interest rates. By hedging away the index exposure, dynamically or statically, you can gain pure exposure to realized volatility, as we shall demonstrate towards the end of this article in the section entitled The Synthetic Capture of Realized Volatility on page 7. But capturing pure volatility is difficult; perfect hedging is impossible and frequent rehedging is expensive, especially close to expiration. Also, as the market moves you may have to roll into new options to maintain a constant volatility sensitivity. Capturing volatility is also risky: there may be unhedgeable index jumps, variation in volatility both foreseen and unforeseen, and varying liquidity. In short, sophisticated investors in pure volatility need complex analytical skills and software, as well as good market instincts, in order to carry out these strategies. For this reason, clients may prefer to buy a contract[4] that delivers pure realized volatility, leaving the mechanics of capture to the dealer providing the contract.

Capturing realized volatility involves trading options whose prices are determined by their implied volatilities. Therefore, the fair value of a derivative contract that delivers pure realized volatility is approximately equal to the level of current implied volatility. Historically, implied index volatility is typically several percentage points higher than past realized volatility over the same period, probably because, as with all risky products, market makers determine prices by factoring in their transactions costs as well some protective risk premium.

USING REALIZED VOLATILITY CONTRACTS

As pointed out, the fair purchase level for realized volatility V is, roughly speaking, the current implied volatility[5] Σ. A **realized volatility contract** is a forward contract on realized volatility V whose delivery price is Σ. The purchaser of the contract with a face value of $1 receives (pays) $1 for every percentage point by which realized volatility V over the life of the contract exceeds (is exceeded by) Σ.

The purchaser of a realized volatility contract will benefit if future realized volatility is greater than current implied levels. He or she is seeking to gain from a belief in future uncertainty, even if that uncertainty ultimately leads to no permanent long-term change in index level.

The seller of a realized volatility contract seeks to gain from a belief that future realized volatility will be appreciably lower than current implied volatility. The owner of a broad portfolio of index options that is long volatility at a variety of market strikes may find this a simple,

4. See *Valuing Contracts with Payoffs Based on Realized Volatility*, Global Derivatives Quarterly Review, Goldman Sachs & Co., July 1996.
5. Volatility mavens will realize that the implied volatilities of individual S&P options vary strongly with strike, expiration and market level. In practice, a more careful view of implied volatility as the cost of purchasing realized volatility may be necessary, taking into account the important effects of the volatility skew and term structure. In the interest of clarity, we ignore most of these subtleties here, though these details obviously effect the engineering and pricing of volatility contracts.

one-stop way to liquidate the portfolio's volatility exposure after a runup in implied volatility levels.

THE ADVANTAGE OF VOLATILITY CONTRACTS

Realized volatility contracts aren't the only way to invest in pure volatility. For example, you can also buy volatility using at-the-money straddles, whose value increases with volatility. But straddles are hybrid: they provide both market and volatility exposure. As the market moves away from the strike, a straddle loses its relatively pure sensitivity to volatility, and evolves into a more complex bet on both market direction and volatility. In order to regain its pure volatility sensitivity, the straddle will have to be rolled at uncertain future market levels and trading costs.

In contrast, realized volatility contracts provide **pure volatility exposure** by design. They provide a straightforward means for clients to accumulate or dispose of volatility as a primary asset, with no related index exposure at all. In the same way as guaranteed exchange rate (Quantos) futures contracts on foreign stock indexes allow investment in foreign markets without exchange-rate risk, so volatility contracts allow index volatility acquisition without index risk.

In principle, volatility contracts can be liquidated at some intermediate date between inception and expiration. As with any forward contract, the value on that date will depend upon both the prevailing implied volatility structure and the volatility realized thus far.

THE SYNTHETIC CAPTURE OF REALIZED VOLATILITY

In this section, we explain how to synthesize pure realized volatility exposure.

First, we need a precise definition of realized volatility V. Let S_0 denote the initial index level at the start of the period, and S_i the closing index value on each subsequent day i of the N days in which the volatility is measured. Then the daily change on the index is $\Delta S_i = (S_{i+1} - S_i)$, the daily return is $(\Delta S_i)/S_i$, and the realized volatility is the square root of the variance of these returns. For an index that moves in a random walk, the mean of the returns is expected to be negligible compared to their individual magnitude. Therefore, we use the following simpler (zero-mean) definition:

$$V = \sqrt{\frac{\sum_{i=0}^{N-1}\left(\frac{\Delta S_i}{S_i}\right)^2}{N-1}} \quad \text{(EQ 1)}$$

Realized volatility is often quoted on an annualized basis by multiplying V by $\sqrt{252}$, the square root of the number of trading days in a year.

To obtain daily volatility exposure, we need to own a position that gains an amount proportional to $((\Delta S)/S)^2$, the daily variance of the index in Equation 1. Note that this expression involves the square of ΔS, and is therefore positive for any index move.

A simple position in one share of the index results in a one-day gain of ΔS, which can be positive or negative, and fails to capture the daily variance. However, a portfolio consisting of an option and its delta hedge produces a one-day gain proportional to $(\Delta S)^2$, always positive.

Figure 4a illustrates the value of an index option position and its delta hedge as the index level varies.The value of an option is convex relative to the hedge: it performs better than the index on up and down moves.

Figure 4b shows the instantaneous gain or loss from owning the option and its hedge after a small shift (ΔS) in the index. You can see that the convexity leads to a gain that is positive whether the index moves up or down. The curve in Figure 4b is a parabola, and the gain is given by $(1/2)\Gamma(\Delta S)^2$, where Γ ("gamma") is a measure of the curvature or convexity of the option, and is a parameter both familiar and of great importance to options traders. The greater the curvature Γ of the track the option rides on, the more sharply the options driver must turn the steering wheel to keep the hedge on track as the index moves, and so, the riskier the position

The quantity $(1/2)\Gamma(\Delta S)^2$ is the gain from an instantaneous index move. The key principle of options valuation is that no free lunch can be obtained by using options. Therefore, if the index actually moves with a realized volatility identical to the implied volatility Σ at which the option was purchased, the gain from small index moves must cancel the loss in option value due to the passage of time[6]. Figure 4c shows

6. As in all options valuation, gains and losses are computed relative to the riskless return.)

this loss due to "time decay"; its magnitude in an instant Δt is given by $(1/2)\Gamma(\Sigma^2 S^2 \Delta t)$. Figure 4d shows the net result: the net gain for the option and its hedge over an instant of time Δt is given by

$$\text{net gain} = (1/2)\Gamma[(\Delta S)^2 - \Sigma^2 S^2 \Delta t] \quad \textbf{(EQ 2)}$$

The first term represents the gain from index moves, the second the loss from time decay during some small time instant Δt. If the index actually moves an amount $\Sigma S \sqrt{\Delta t}$ during the instant Δt, consistent with its implied volatility, there is neither gain nor loss from the hedged option, and the hedged position breaks even.

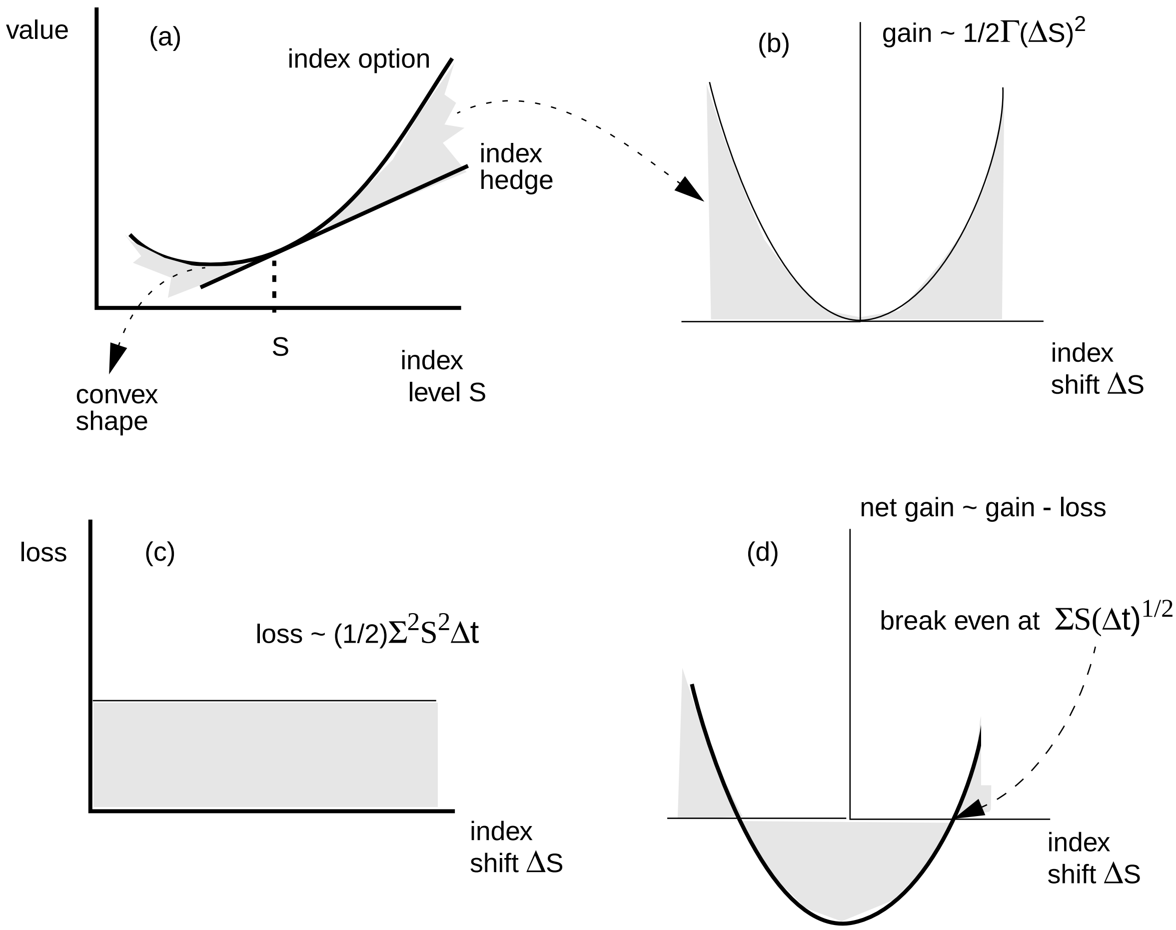
Equation 2 shows that the gain on the hedged position is proportional to $\Gamma(\Delta S)^2$. Recall from Equation 1 that a gain proportional to $(\Delta S/S)^2$ will capture the daily realized volatility. We can achieve this by owning a delta-hedged options position whose curvature Γ is proportional to $1/S^2$, the inverse of the square of the index level. *The daily trading gain of the hedged position is then equal to the realized volatility each day, less the initial implied volatility.*

There are both dynamic and static approaches to constructing an options position with $\Gamma \sim 1/S^2$. In the dynamic method, at each index level S, you continually maintain a portfolio consisting of a quantity of $1/S^2$ options, each individually providing a zero Δ and a constant Γ. As an example, you can use at-the-money forward straddles, which instantaneously have the appropriate Δ and constant Γ. As the index moves, you have to keep rolling to new at-the-money straddles.

The static method requires buying an options portfolio whose Γ is guaranteed to be $1/S^2$ at all market levels, and then delta-hedging it. This static portfolio can be shown to be an index derivative contract whose payoff at expiration is proportional to the natural logarithm of the index[7]. Since this "log contract" is unavailable in the listed market, you can approximate it by a combination of ordinary listed options that has approximately the same payoff. Static or dynamic, the method we use in practice depends on prevailing options spreads, estimates of liquidities and trading costs.

7. See Neuberger [1994] and *Valuing Contracts with Payoffs Based on Realized Volatility*, Global Derivatives Quarterly Review, Goldman Sachs & Co., July 1996.

FIGURE 4. (a) The value of an index option and its hedge. (b) The quadratic gain of the delta-hedged contract as a function of shift in index level. (c) The loss from time decay over time Δt. (d) The net gain on the hedged position.



If you are interested in volatility as an asset, it may be simplest to buy a realized volatility contract from a trading desk that can synthesize it. At present, dealers can provide forwards on volatility in the form of realized volatility contracts. Over time, as both the analytical and trading technology mature, we expect to see the evolution of volatility options and related volatility derivatives.

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