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The Local Volatility Surface

Unlocking the Information in Index Option Prices

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SUMMARY

If you examine the structure of listed index options prices through the prism of the implied tree model, you observe the local volatility surface of the underlying index.

In the same way as fixed income investors analyze the yield curve in terms of forward rates, so index options investors should analyze the volatility smile in terms of local volatilities.

In this report we explain the local volatility surface, give examples of its applications, and propose several heuristic rules of thumb for understanding the relation between local and implied volatilities. In essence, the model allows the extraction of the fair local volatility of an index at all future times and market levels, as implied by current options prices. We use these local volatilities in markets with a pronounced smile to measure options market sentiment, to compute the evolution of standard options implied volatilities, to calculate the index exposure of standard index options, and to value and hedge exotic options. In markets with significant smiles, all of our results show large discrepancies from the results of the standard Black-Scholes approach.

Investors who buy or sell standard index options for the exposure they provide, as well as market participants interested in the fair price of exotic index options, should find interest in the deviations we predict from the Black-Scholes results.

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THE IMPLIED VOLATILITY SURFACE

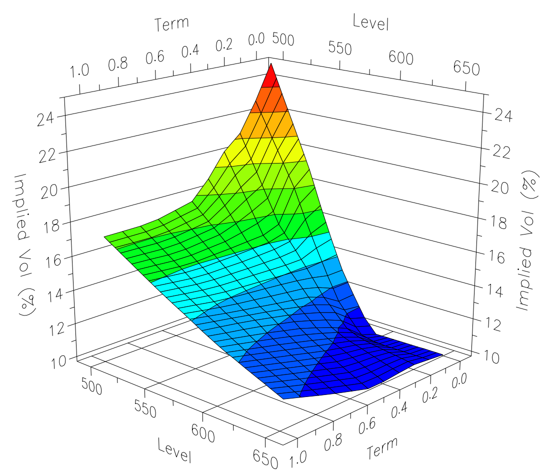
The market implied volatilities of standard equity index options commonly vary with both option strike and option expiration. This structure has been a significant and persistent feature of index markets around the world since the 1987 crash. Figure 1 shows a typical implied volatility surface illustrating the variation of S&P 500 implied volatility with strike and expiration on September 27, 1995[1]. This surface, commonly called the “volatility smile,” changes shape from day to day, but some general features persist.

The strike structure. At any fixed expiration, implied volatilities vary with strike level. Almost always, implied volatilities increase with decreasing strike – that is, out-of-the-money puts trade at higher implied volatilities than out-of-the-money calls. This feature is often referred to as a “negative” skew.

The term structure. For any fixed strike level, implied volatilities vary with time to expiration. Often, long-term implied volatilities exceed short-term implied volatilities.

In brief, there is a unique and generally different implied volatility associated with any specific strike and expiration.

FIGURE 1. **The implied volatility surface for S&P 500 index options as a function of strike level and term to expiration on September 27, 1995.**



1. Liquid listed options have discrete strikes and expirations, and so we have interpolated between them to create a continuous surface.

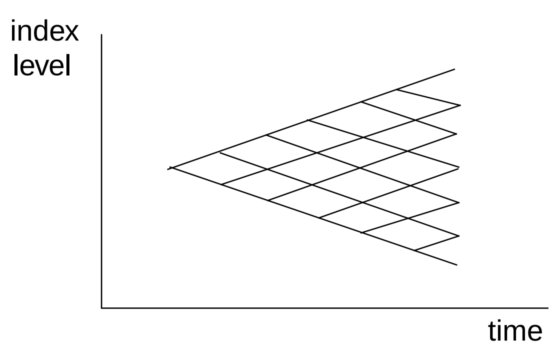
THE IMPLIED VOLATILITY SURFACE BELIES THE BLACK-SCHOLES MODEL

Each implied volatility depicted in the surface of Figure 1 is the Black-Scholes implied volatility, Σ, the volatility you have to enter into the Black-Scholes formula to have its theoretical option value match the option's market price. Σ is the conventional unit in which options market-makers quote prices. What does the varying volatility surface for Σ tell us about the model and the world it attempts to describe?

The primary feature of the Black-Scholes [1973] model of options valuation theory is that it is *preference-free:* since options can be hedged, their theoretical values do not depend upon investors' risk preferences. Therefore, an index option can be valued *as though* the return on the underlying index is riskless.

A secondary feature of the theory is its assumption that returns on stocks or indexes evolve normally, with a *local volatility* σ *that remains constant over all times and market levels*. Figure 2 contains a schematic representation of the index evolution in a binomial tree framework[2]. The constant index level spacing in the figure corresponds to the assumption of constant local return volatility. These two features leads to the Black-Scholes formula $C_{BS}(S, \sigma, r, T, K)$ for a call on an index at level S with a volatility σ, with strike K and time to expiration T, when the riskless interest rate is r.

FIGURE 2. **A schematic representation of index evolution in the Black-Scholes model.**



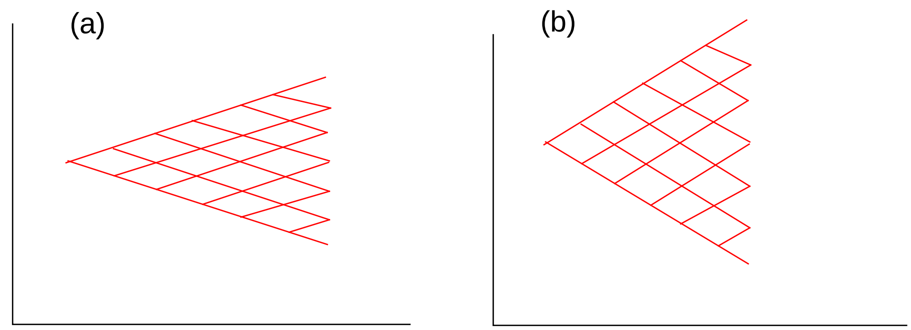
2. In mathematical terms, the evolution over an infinitesimal time dt is described by the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

where S is the index level, μ is the index's expected return and dZ is a Wiener process with a mean of zero and a variance equal to dt.

When options market makers quote an implied volatility Σ for an option of a given strike and expiration, they are specifying the future local volatility $\sigma = \Sigma$ that you must enter into the Black-Scholes formula to obtain the market price for the option, *assuming that* σ *stays constant over all times and market levels*. By quoting two different implied volatilities Σ_1 and Σ_2 for two different options they are attributing two different constant local volatilities to the same underlying index, as illustrated in Figure 3. This belies the Black-Scholes picture. There is only one index underneath all the options and, for consistency, *it can only have one implied evolution process in equilibrium*. The market is using the Black-Scholes formula as a mechanism for conveying information about its equilibrium prices, but, in the act of quoting prices, belying the assumptions of the model. As we shall point out later, this closely resembles the situation in bond markets, where traders quote bond prices by their yield to maturity, but calculate them by using forward rates.

FIGURE 3. Schematic representation of the binomial index trees corresponding to two options with different Black-Scholes implied volatilities: (a) long-term option with low implied volatility; (b) short-term option with high implied volatility.



THE LOCAL VOLATILITY SURFACE

There is a simple extension to the strict Black-Scholes view of the world that can achieve consistency with the index market's implied volatility surface, without losing many of the theoretical and practical advantages of the Black-Scholes model.

Rational market makers are likely to base options prices on their estimates of future volatility[3]. To them, the Black-Scholes Σ is, roughly speaking, a sort of estimated *average* future volatility of the index during the option's lifetime. In this sense, Σ is a *global* measure of volatility, in contrast to the *local* volatility σ at any node in the tree of index evolution. Until now, theory has tended to disregard the difference between Σ and σ. Our path from now on is to accentuate this difference, and deduce the market's expectations of local σ from the values it quotes for global Σ.

The variation in market Σ indicates that the average future volatility attributed to the index by the options market depends on the strike and expiration of the option. A quantity whose average varies with the range over which it's calculated must itself vary locally. The variation in Σ with strike and expiration implies a variation in σ with future index level and time. In other words, the implied volatility surface suggests an obscure, hitherto hidden, local volatility surface.

Assuming that options prices are efficient, we can treat all of them consistently in a model that simply abandons the notion that future volatilities will remain constant. Instead, we extract the market's consensus for future local volatilities $\sigma(S,t)$, as a function of future index level S and time t, from the spectrum of available options prices as quoted by their implied Black-Scholes volatilities. Schematically, we replace the regular binomial tree of Figure 2 by an *implied tree*[4], as shown in Figure 4. Derman and Kani [1994] and, separately, Dupire [1994] have shown that, if you know standard index options prices of all strikes and expirations, then in principle you can uniquely determine the local volatility surface function $\sigma(S,t)$. A similar, though not identical, approach has been taken by Rubinstein [1994].

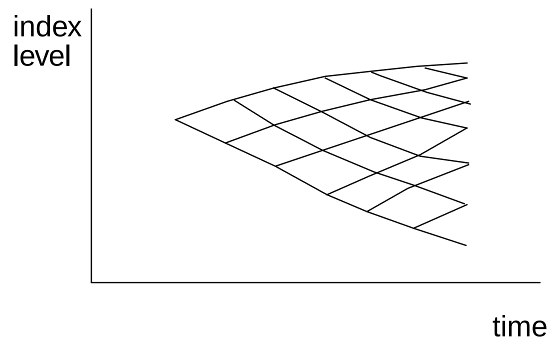
3. This is not to say that this is the only important factor. In particular, traders will also take hedging costs, hedging difficulties and liquidity into account, to name only a few additional variables.

4. Our new model replaces the evolution equation in footnote 2 by

$$\frac{dS}{S} = \mu dt + \sigma(S, t) dZ$$

where $\sigma(S,t)$ is the local volatility function whose magnitude depends on both the index level S and the future time t.

FIGURE 4. **The implied binomial tree.**

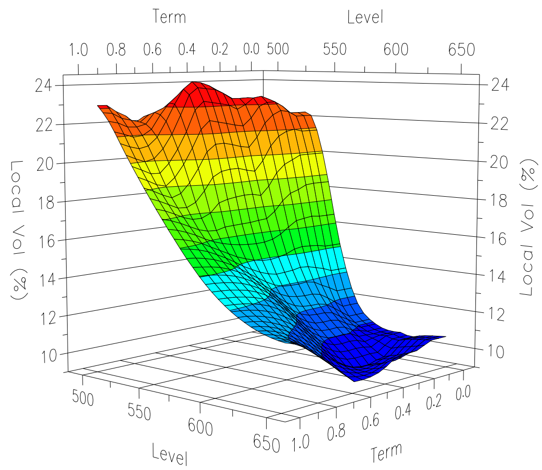


In essence, our model assumes that index options prices (that is, implied volatilities) are driven by the market's view of local index volatility in the future. We have shown that you can theoretically extract this view of the local volatility $\sigma(S,t)$ from standard options prices. Readers familiar with the habits of options traders will realize that thinking about future volatility is an intrinsic part of their job. Many traders intuitively deduce future local volatilities from options prices. Our model provides a more quantitative and exact way of accomplishing this. Figure 5 displays the local volatility surface corresponding to the implied volatility surface of Figure 1.

Our approach preserves many of the attractive facets of the Black-Scholes model, while extending it to achieve consistency with market options prices. The great advantage of the Black-Scholes model in a trading environment is that it provides preference-free pricing. Its inputs are current index levels, estimated dividend yields and interest rates, most of which are determined and well-known. All the model asks of a user is *one* implied volatility, which it translates into an option price, an index exposure, and so on.

The implied tree model preserves this quality. In brief, all it asks of a user is the implied Black-Scholes volatility of *several* liquid options of various strikes and expirations. The model fits a consistent implied tree to these prices, and then allows the calculation of the fair values and exposures of all (standard and exotic) options, consistent with all the initial liquid options prices. Since traders know (or have opinions about) the market for current liquid standard options, this makes it especially useful for valuing exotic index options consistently with the standard index options used to hedge them.

FIGURE 5. The local volatility surface corresponding to the implied volatilities of Figure 1.



In addition to the variation of the local volatility surface, there are other factors that can complicate index options pricing, and so contribute to a non-flat implied volatility surface. For example, volatility itself has a stochastic component, and markets sometimes jump in a manner inconsistent with the continuous evolution of implied tree models. In addition, the Black-Scholes model ignores the effects of transactions costs. All of these phenomena can contribute to the smile. By using the implied tree model, we are assuming that the variation of local volatility with market level and time is the dominant contribution to the smile, and that other effects are less important. We assume that options market makers think most about the level volatility may take in the future. In this article we try to fully exploit this small change in the Black-Scholes framework, otherwise preserving the attractive and useful features of the model. We prefer to use the more complex and less preference-free models involving jumps and stochastic volatility only when our simpler approach becomes inadequate.

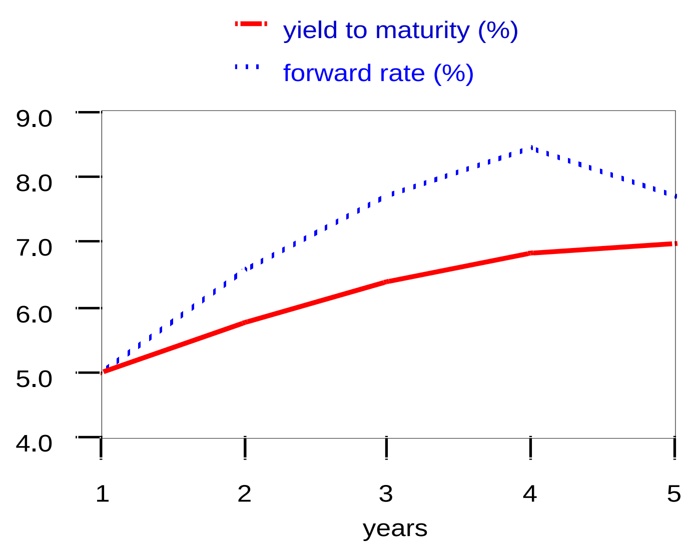
THE ANALOGY BETWEEN FORWARD RATES AND LOCAL VOLATILITIES

The implied-tree approach to modeling the volatility smile stresses the use of local volatilities extracted from implied volatilities. Our incentive to analyze value in terms of local quantities rather than global averages is analogous to a similar historical development in the analysis of fixed income securities more than a generation ago.

Suppose you know the quoted yield to maturity for all on-the-run Treasury bonds, and you want to value an off-the-run ("exotic") Treasury bond whose coupon and maturity differ from those of any on-the-run bond. Figure 6 displays the coupons, yields and one-year forward rates for a hypothetical set of Treasury bonds. What yield to maturity should you use to value the exotic Treasury?

FIGURE 6. **Yields to maturity and one-year forward rates for a hypothetical Treasury bond market. Coupons are paid annually. All rates are compounded annually.**

maturity	coupon	price	yield	forward rate
1 year	5.00%	100	5.00%	5.00%
2	5.50	99.5	5.77	6.59
3	6.00	99.0	6.38	7.72
4	6.25	98.0	6.84	8.43
5	6.50	98.0	6.99	7.71



There is a close analogy between the dilemma in trying to value an off-the-run Treasury bond by picking the “correct” yield to maturity and the dilemma in trying to value an exotic option by picking the “correct” implied volatility. In the Treasury bond market, each bond has its own yield to maturity. The yield to maturity of a bond is actually the *implied constant forward discount rate* that equates the present value of a bond’s coupon and principal payments to its current market price. Similarly, in the index options market, each standard option has its own implied volatility, which is the *implied constant future local volatility* that equates the Black-Scholes value of an option to its current market price.

For off-the-run bond valuation, the correct and time-honored approach is to eschew yield to maturity, and instead use the on-the-run yield curve to deduce forward rates, and then use these forward rates to discount the coupons of off-the-run bonds. Implied trees take a similar approach to exotic options. They avoid implied volatility, and instead use the volatility surface of liquid standard options to deduce future local volatilities. Then, they use these local volatilities to value all exotic options. We illustrate the similarities in Table 1.

TABLE 1. The analogy between forward rates and local volatilities

Aim	To value an *off-the-run Treasury bond:*	To value an *exotic index option:*
Old Approach	You used *simple yield-to-maturity* to discount all future coupons and principal.	You used *simple implied volatility* to calculate the risk-neutral probability of future payoff.
New Approach	Use zero-coupon *forward rates* constructed from liquid Treasury bond prices to discount all future coupons and principal.	Use local *volatilities* constructed from liquid standard options prices to calculate the risk-neutral probability of future payoff.

In the Treasury bond market, this approach makes sense if you’re a hedger or arbitrageur. In that case, you’re interested in the value of an off-the-run bond relative to the on-the-runs. Forward rates are the appropriate way to determine relative value at the present time, no matter what happens later. On the other hand, if you’re a speculator you’re more interested in whether forward rates are good predictors of future rates, and arbitrage pricing is less important. Similarly, in the equity index options market, local volatilities are the appropriate way to determine the value of an exotic option relative to standard options, no matter what future levels volatility takes.

You can lock in forward rates by buying a longer-term bond and selling a shorter-term bond so that your net cost is zero. Analogously, you can lock in forward (local) volatility by buying a calendar spread and selling butterfly spreads with a zero net cost.

USING THE LOCAL VOLATILITY SURFACE

Implied tree models of the skew are dynamical. They postulate a process for future index evolution in which the local volatility function $\sigma(S,t)$ depends on S and t. The function $\sigma()$ is determined by the constraint that the fair value of all standard options calculated from this evolution process match current options market prices. Once $\sigma(S,t)$ is fixed, all future index evolution is known, and you can calculate a no-arbitrage value for any derivative security in a manner consistent with current options prices.

In this section we point out several areas where implied tree models lead to significantly different, and sometimes counter-intuitive, results when compared with the Black-Scholes model.

Obtaining the local volatilities

You need the following information to extract the local volatility surface at any instant:

1. the current value of the index;
2. the current (zero-coupon) riskless discount curve;
3. the values and *ex* dates of future index dividends; and
4. liquid standard options prices for a range of strikes and expirations, or (more commonly) their Black-Scholes implied volatilities.

Figure 7 shows the data entry window of a Goldman Sachs program for calculating local volatilities. The array of standard options' implied volatilities has been displayed as an implied volatility surface.

You can apply this procedure to any options market with good pricing information for options of various maturities and strikes. Figure 8a displays the local volatility surface of the S&P 500 index on Dec. 19, 1995, as extracted from the implied volatilities of Figure 7 using an Edgeworth expansion technique due to Zou [1995]. Figure 8b shows the Nikkei 225 index local volatility surface on Dec. 2, 1994. The negative skew in both the S&P and Nikkei markets produces surfaces for which local volatility increases as market levels decrease.

These local volatilities represent the collective expectation of options market participants, assuming the options prices are fair. It's important to note that these local volatilities are not necessarily good predictors of future realized volatility, just as forward interest rates are not necessarily good predictors of future rates. Just as investors can use long/short bond portfolios to lock in forward interest rates, so they can use options to lock in future local volatilities.

FIGURE 7. Inputs to a program for calculating local volatilities. The plot shows the implied volatility surface for the S&P 500 index on October 10, 1995. Estimated future dividends of the index are not displayed.

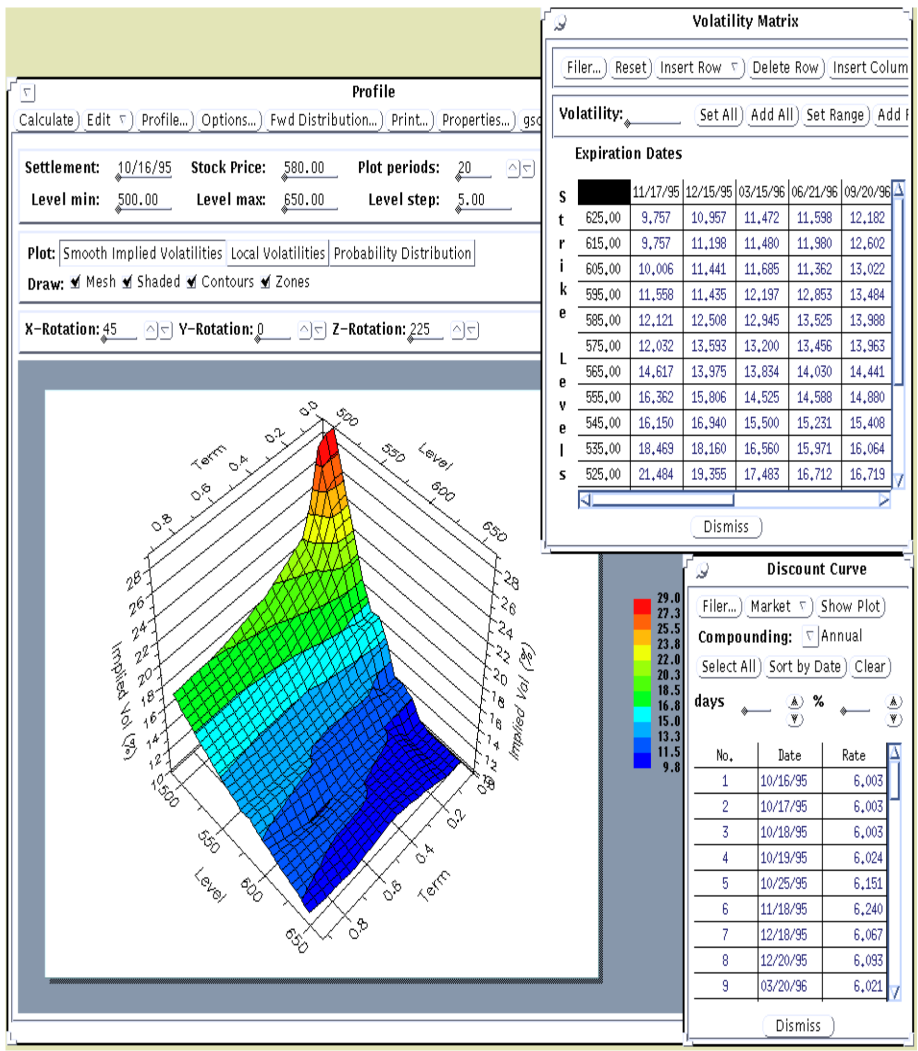
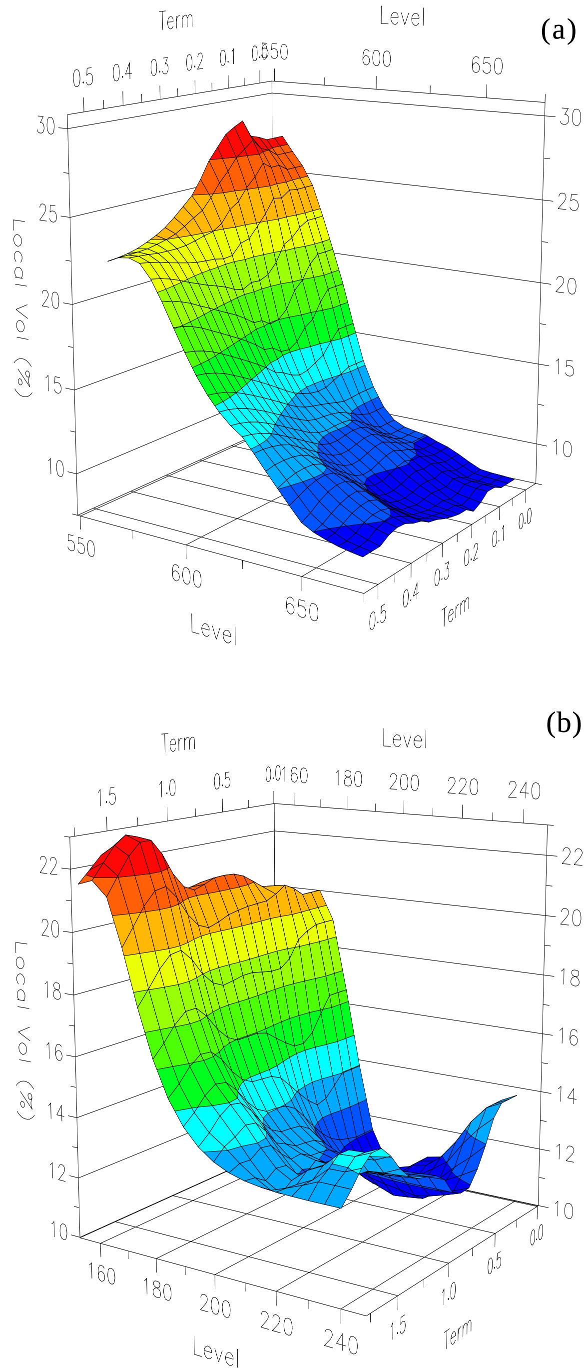


FIGURE 8. (a) The local volatility surface for the S&P 500 on Dec. 19, 1995. (b) The local volatility surface for the Nikkei 225 on Dec. 2, 1994. The Nikkei index level is shown in multiples of 100.



The correlation between index level and index volatility

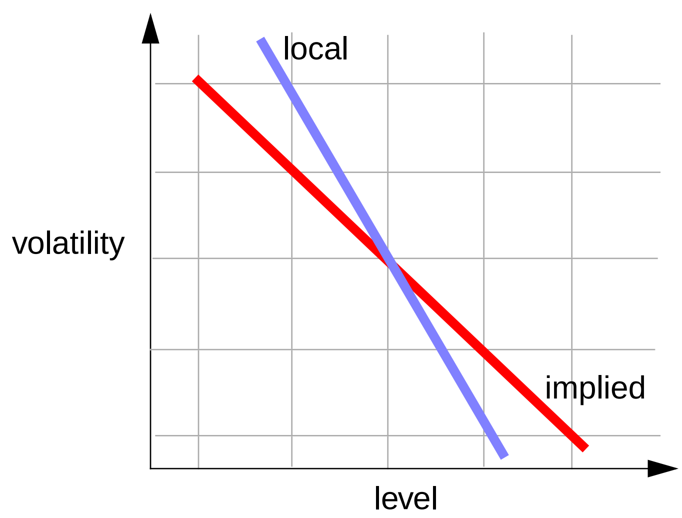
The local volatility surface indicates the fair value of local volatility at future times and market levels. The most striking feature of Figure 8 is the systematic decrease of local index volatility with increasing index level. This implied correlation between index level and local volatility is essentially responsible for all of the qualitative features of our results below.

The variation in S&P 500 local volatility displayed in Figure 8a is generally greater than the variation in implied volatility in Figure 7 that produced it. For skewed options markets, we note the following heuristic rule[5]:

> **Rule of Thumb 1:** Local volatility varies with market level about twice as rapidly as implied volatility varies with strike.

Figure 9 illustrates this relationship. For theoretical insight, see the Appendix, and also Kani and Kamal [1996].

FIGURE 9. **In the implied tree model, local volatility varies with index level approximately twice as rapidly as implied volatility varies with strike level.**



5. The three rules of thumb that appear below apply to short and intermediate term equity index options, where the correlation between index level and volatility is most pronounced and the assumption of approximately linear skew seems to be good. For longer term options, other factors, such as stochastic volatility or volatility mean reversion, may start to blur the effects of correlation that we have encapsulated in these three rules.

Implied distributions and market sentiment

If you can estimate future index dividend yields and growth rates, you can use the local volatility surface to simulate the evolution of the index to generate index distributions at any future time. Figure 10 shows the end-of-year S&P 500 distributions implied by liquid options prices during mid-July 1995, a turbulent time for the U.S. equity market. In generating these distributions we have assumed an expected annual growth rate of 6% and dividend yield of 2.5% per year.

On Monday, July 17, the S&P 500 index closed at a record high of 562.72. On Tuesday, July 18, the Dow Jones Industrials Index dropped more than 50 points and the S&P closed at 558.46. On Wednesday, July 19, the Dow Jones fell about another 57 points, and the S&P closed at 550.98. The shift in market sentiment during these three days is reflected in the changing shapes of the distributions. For instance, a shoulder at the 550 level materialized on July 18, indicating a more negative view of the market. By July 19, a peak at the 480 level became apparent.

Investors whose views of future market distributions differ from that implied by options prices can take advantage of the differences by buying or selling options.

The future evolution of the smile

Once fitted to current interest rates, dividend yields and implied volatilities, the implied tree model produces a tree of future index levels and their associated fair local volatility, as implied by options prices. Figure 11 displays a schematic version of the implied tree for a negatively skewed market. with its origin at the time labeled current. Assuming the market's perception of local volatility remains unchanged as time passes and the index moves, we can use these local volatilities to calculate the dependence of implied volatility on strike at future times. If, at some time labeled later in Figure 11, the index moves to either of the levels labeled up or down, the evolution of the index is described by the subtrees labeled up or down. This is valid provided no new information about future volatility, other than a market level move, has arrived between the time the initial tree was built and the time at which the index has moved to the start of a new subtree.You can use the up or down tree to calculate fair values for options of all strikes and expirations at time later. You can then convert these prices into Black-Scholes implied volatilities, and so compute the fair future implied volatility surfaces and skew plots.

FIGURE 10. Implied S&P 500 distributions on Dec. 31, 1995, based on S&P 500 implied volatilities on July 17, 18 and 19, 1995. We assume a growth rate of 6% and dividend yield of 2.5% to year end.

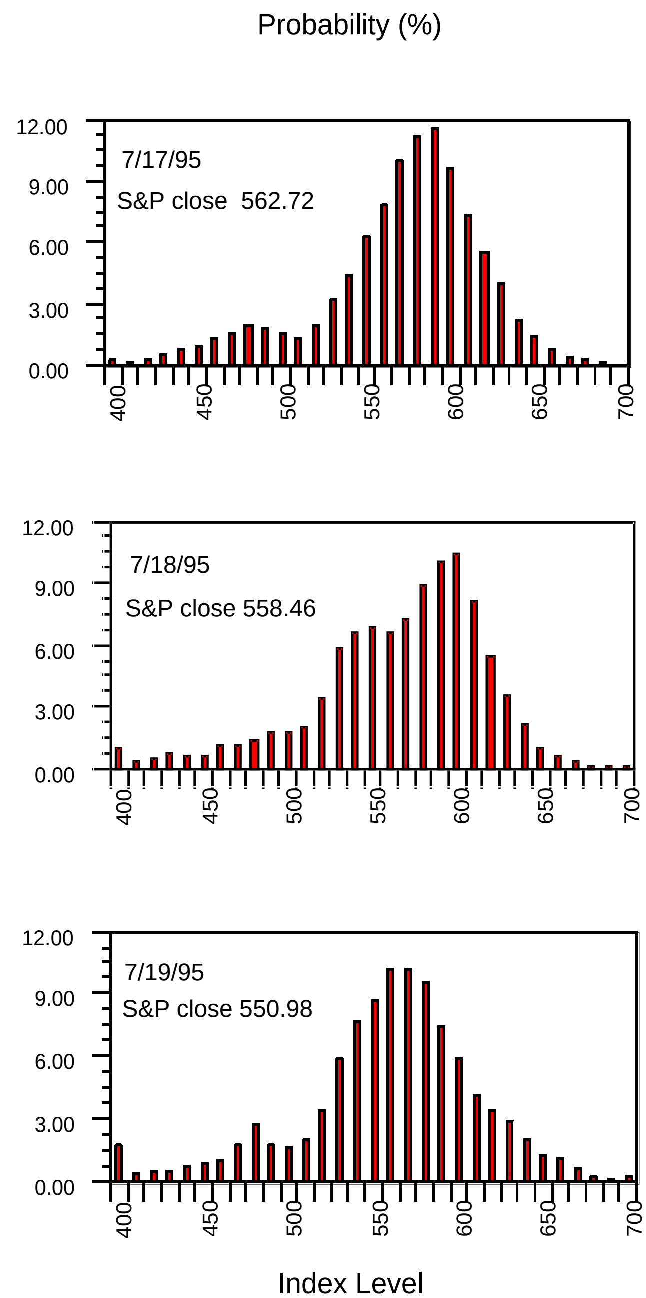
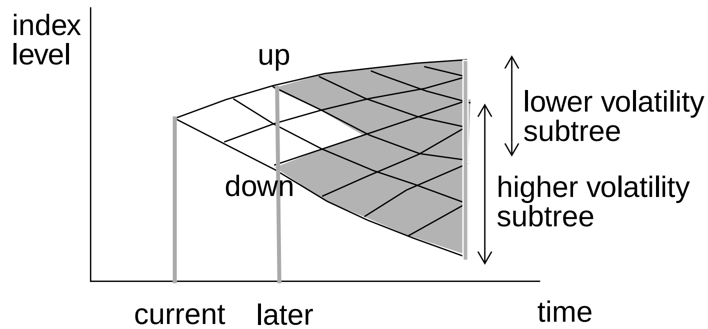


FIGURE 11. Schematic illustration of the implied tree and the trees contained within it in a negatively skewed market. Larger spacing depicts higher local volatility.



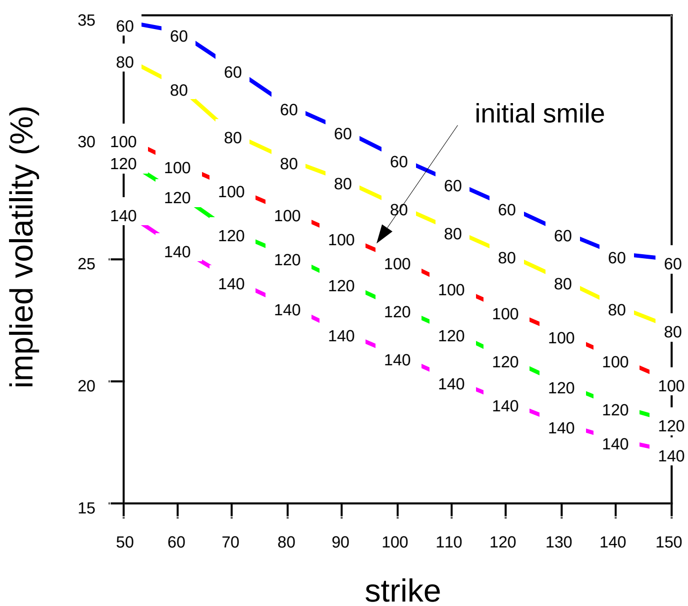
Let's look at an example for a negatively skewed index like the S&P 500. To be specific, consider standard options on an index whose current level is 100, with a riskless interest rate of 7% and a dividend yield of 2%. We assume annual at-the-money implied volatility to be 25%, with a hypothetical constant negative skew of one volatility point decrease for every ten-point increase in strike. For simplicity we assume that all these rates, yields and volatilities are independent of maturity or expiration – that is, all term structures are flat. Figure 12 shows the fair implied volatility skew for one-year options six months after the initial implied tree was constructed. You can see that, for negative skews, the implied volatility of an option with any particular strike tends to move down as the market moves up.

Here's another heuristic rule for implied tree models:

> **Rule of Thumb 2:** The change in implied volatility of a given option for a change in market level is about the same as the change in implied volatility for a change in strike level.

For example, if the skew is such that a one-point change in strike leads to a half-percentage-point change in implied volatility, then so does a one-point change in market level. If you know the observed skew at a fixed market level, then you know what happens to a given option's value when the market moves. For further elaboration, see the Appendix.

FIGURE 12. Evolution of the smile: the smile at a variety of index levels, assuming an initial index level of 100. The line labeled 100 is the initial smile. Other lines represent the implied tree's fair skews at different market levels six months in the future, as indicated by the corresponding labels.

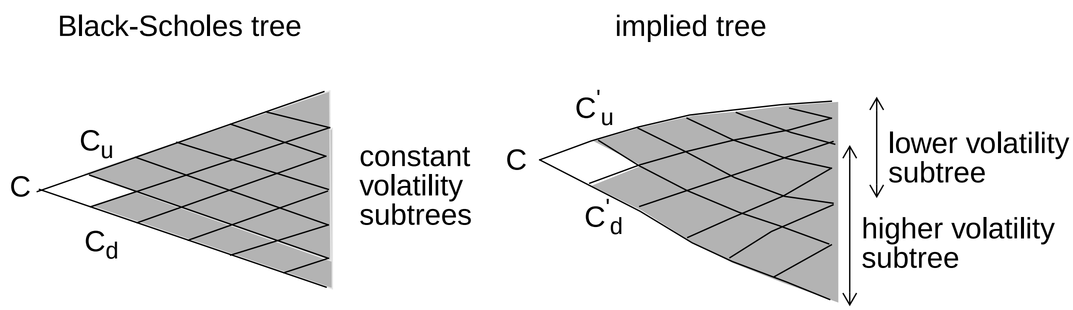


The skew-adjusted index exposure of standard options

Implied tree models are constrained to fit current liquid standard options prices. As we illustrated in Figure 11, for negatively skewed volatility markets like the S&P 500, local volatility falls as the index rises. Now, implied volatility is a global average over local volatilities. Therefore, for any particular option, implied Black-Scholes volatility is anti-correlated with the index level, falling as the index rises and rising as it falls. We illustrate this point in Figure 13, where the evolution of the initial value of a call, C, as the index moves up or down, is shown in both the Black-Scholes and implied tree models

In the notation of Figure 13, the index exposure of the call in the Black-Scholes model is proportional to $C_u - C_d$. In the implied tree model the exposure is proportional to $C'_u - C'_d$. But the negative correlation of volatility with index level in the implied tree means that $C'_u < C_u$ and $C'_d > C_d$, so that $(C'_u - C'_d) < (C_u - C_d)$. The exposure of the call in the implied tree model with negative skew is consequently

FIGURE 13. The delta exposure of a call option in the Black-Scholes model and the implied tree model. $C_{u(d)}$ **denotes the value of the call** C **after an upward(downward) index move in a constant volatility.** $C'_{u(d)}$ **denotes the value of the same call in the implied tree model.**



lower than it would have been in a Black-Scholes world with flat volatility. The implied-tree exposure of a put under the same circumstances is also lower (that is, more negative) than the corresponding Black-Scholes exposure.

In the implied tree model, a rise in index level influences the value of a call option in two ways. First, the call moves deeper into the money. Second, the volatility of the call decreases because of the correlation between index and local volatility. With this in mind, you can use the Black-Scholes formula and Rule of Thumb 2, as explained in the Appendix, to derive the following heuristic rule for the option's exposure:

Rule of Thumb 3: The correct exposure Δ of an option is approximately given by

$$\Delta = \Delta_{BS} + V_{BS} \times \beta$$

where Δ_{BS} is the Black-Scholes exposure (in dollars per index point), V_{BS} is the Black-Scholes volatility sensitivity (in dollars per volatility point), and β is the observed sensitivity of implied volatility to strike level (in volatility points per strike point). β is negative in options markets where implied volatility decreases with strike.

Table 2 illustrates the effect of the skew on a call's index exposure using the above rule of thumb. The market parameters chosen correspond roughly to those of the S&P 500. The rule-of-thumb exposure is 48% of the underlying index, 11 percentage points lower than the naively-calculated Black-Scholes exposure of 59%. This is a significant difference.

TABLE 2. The effect of the smile on the index exposure of a call option (See Rule of Thumb 3)

Market		Call Option		Exposure	
Index	600	Strike	600	Δ_{BS}	0.59
Dividend yield	3%	Expiration	1 year	$V_{BS} \times \beta$	-0.11
Volatility (a-t-m)	13%	Black-Scholes value	38.4	Rule-of-thumb Δ	0.48
Skew slope β	-0.05*	Black-Scholes volatility sensitivity V	2.23**		
Riskless rate	6%				

* in percentage points per index strike point.
** in dollars per volatility point.

The theoretical value of barrier options

The theoretical value of a barrier option depends on the risk-neutral probability of the index being in-the-money at expiration, but not having crossed the barrier during the option's lifetime. This probability is very sensitive to volatility levels in general, and to the volatility skew in particular. The traditional, and widely used, analytical formula [Merton 1973] for barrier options applies only in the absence of skew, and is not a good guide when appreciable skews exist.

We illustrate this examining at a one-year up-and-out European-style call option on an index with strike at 100% of the index level and barrier at 130%. Figure 14 shows the hypothetical implied volatility skew we use to illustrate the valuation. We also assume a riskless annual interest rate of 5% and zero dividend yield.

FIGURE 14. A hypothetical volatility skew for options of any expiration. We assume a constant riskless discount rate of 5% and a zero dividend yield. The arrows show the strike (100) and barrier (130) level of the up-and-out option under consideration.

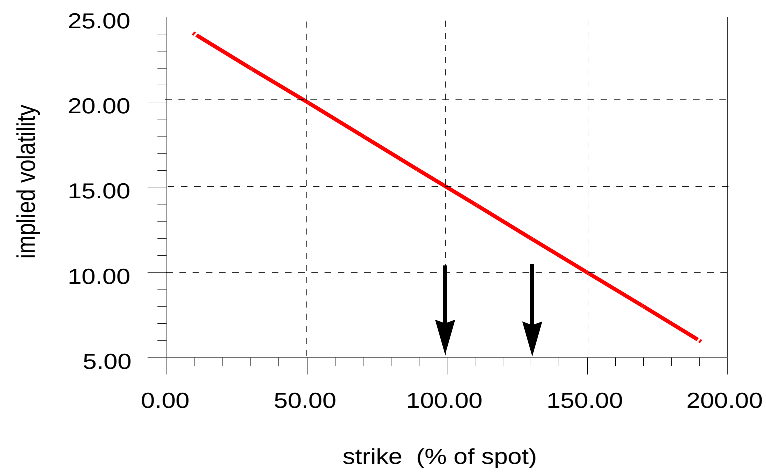


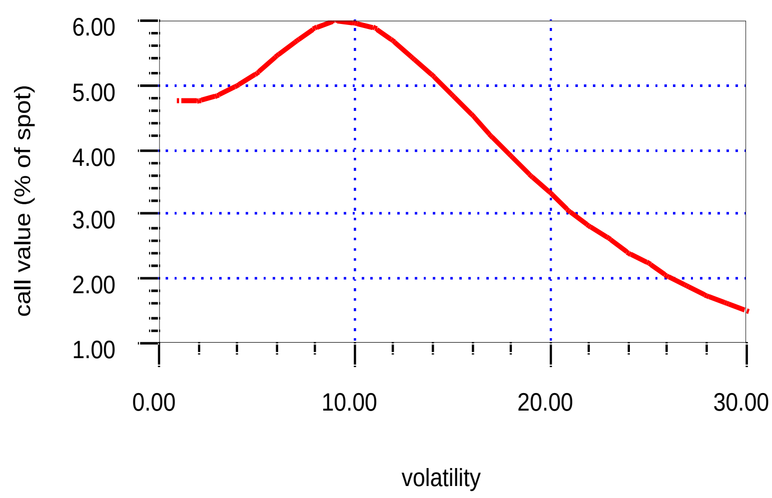
Figure 15 shows the variation in the theoretical value of the knock-out call as a function of implied volatility in a world with no skew. The value of the call peaks at 5.99% when the volatility is about 9.6%. Any further increase in volatility causes a decrease in call value because the additional likelihood of knockout before expiration outweighs the additional probability of finishing in-the-money.

According to our implied tree model, constrained to fit the skew of Figure 14, the up-and-out option is worth 6.46% of the current index value. This is greater than any value in the skewless world of Figure 15. There is *no single value* of volatility in a skewless model that can account for the implied tree call value. No amount of intuition can lead you to guess the “right” volatility value to insert into the flat-volatility “wrong” model to reproduce the “right” knockout call value caused by the skew.

Valuing path-dependent options using Monte Carlo simulation

Path-dependent options contain embedded strikes at multiple market levels, and are consequently sensitive to local volatility in multiple regions. When implied volatility varies with strike or expiration, no single constant volatility is correct for valuing a path-dependent option. However, you can simulate the index evolution over all future market levels and their corresponding local volatilities to calculate

FIGURE 15. Theoretical value of an up-and-out, at-the-money, European-style call option as a function of volatility in a flat-volatility world. Strike = 100; barrier = 130; dividend yield = 0; annual discount rate = 5%.

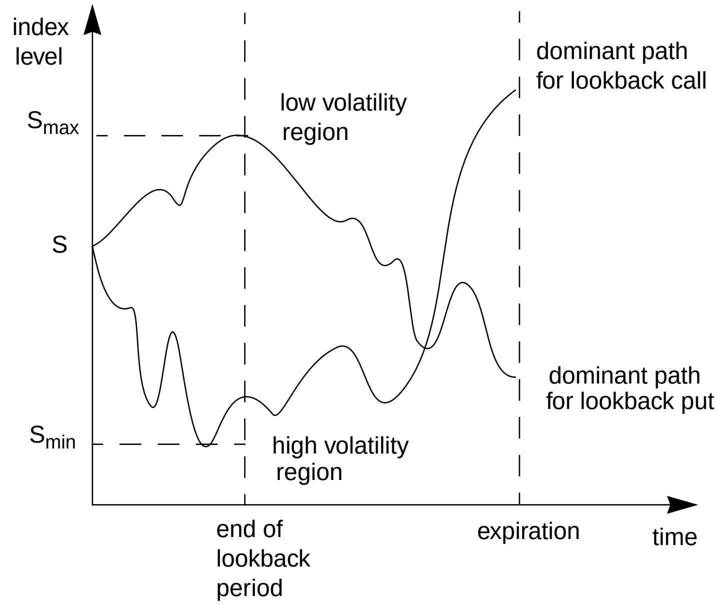


the fair value of the option. We illustrate this approach for a European-style lookback call and put. The method is general and can be applied to Asian options, as well as other path-dependent derivative instruments.

Now consider a one-year lookback call or put with a three-month lookback period on the strike. The call and put payoffs at expiration are $max[S' - S_{min}]$ and $max[S_{max} - S']$ respectively, where S' is the terminal index level and S_{min} (S_{max}) is the lowest (highest) level the index reaches during the first three months of the option's life. We value the securities by simulating index paths whose local volatilities are extracted from the relevant implied volatility smile. For each path we calculate the present value of the eventual payoff of the lookback call, averaging over all paths to obtain the current value of the call. We duplicate this procedure to value the lookback put.

Figure 16 shows the dominant index evolution paths – the paths that contribute the most value – to the lookback calls and puts. A dominant path for a lookback call sets a low strike minimum during the first three months, and then rises to achieve a high payoff. The theoretical value of the call is determined by (i) the likelihood of setting a low strike, and then, the strike having been fixed and the lookback

FIGURE 16. Dominant paths contributing to the value of a lookback call and put. The local volatilities are negatively skewed.



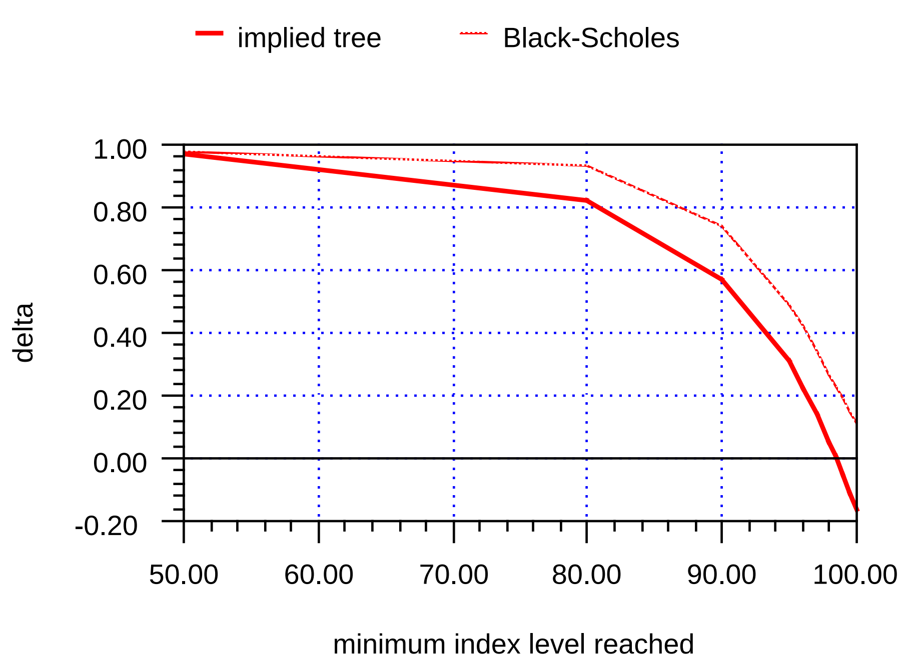
option having become a standard option, (ii) the subsequent volatility of the index. Similarly, a dominant path for a lookback put sets a high initial strike and then drops. Its value is determined by the likelihood of a high strike and the subsequent index volatility.

In the implied tree model with a negative volatility skew, higher strikes and index levels correlate with lower index volatility. Therefore, the dominant path for a lookback call is more likely to have an advantageously low strike S_{min}, and a high subsequent volatility. Conversely, the dominant path for a lookback put is more likely to have a disadvantageously low strike S_{max}, and a low subsequent volatility. Therefore, in a negatively skewed world, lookback puts are worth relatively less, and lookback calls more. When options values are quoted in terms of their Black-Scholes (unskewed) implied volatilities, lookback calls will have higher implied volatilities than lookback puts.

For illustration, we now assume a hypothetical index level of 100, a dividend yield of 2.5%, and a riskless rate of 6% per year. The index has a negative skew that is assumed to be independent of expiration: at-the-money implied volatility is 15%, and decreases by 3 percentage points for each increase of 10 index strike points. Using Monte Carlo simulation, we find the fair value of the lookback call to be 10.8% of the index, and the value of the lookback put to be 5.8%. In the framework of an unskewed, Black-Scholes index, these values correspond to an implied volatility of 15.6% for the lookback call and 13.0% for the lookback put.

You can use the same method to calculate the deltas of lookback options. Figure 17 compares the implied-tree deltas with the Black-Scholes deltas[6] for the one-year lookback call described above, for a range of minimum index levels previously reached when the index level is currently at 100. The Black-Scholes deltas are calculated at the Black-Scholes implied volatility of 15.6% that matches the value obtained by Monte Carlo simulation value over the skewed local volatilities.

FIGURE 17. **The delta of a one-year call with a three-month lookback period that has identical prices in the implied tree model and the Black-Scholes world with no skew. The current market level is assumed to be 100.**



6. The expression “Black-Scholes delta” is shorthand for the delta in a Black-Scholes world – that is, a world where local volatility is constant, independent of future time and future index level. Similarly, “Black-Scholes implied volatility” is shorthand for the constant local volatility in a Black-Scholes world that results in a theoretical value that matches the dollar value of the option.

Note that the delta of the lookback call is always lower in the implied tree model than in the Black-Scholes model. This mismatch in model deltas occurs because, in the implied tree model, the option's sensitivity to volatility also contributes to its index exposure through the correlation between volatility and index level (see Rule of Thumb 3). The mismatch is greatest where volatility sensitivity is largest, that is, where the minimum index level is close to the current index level. The mismatch is correspondingly smallest when the lowest level previously reached is much lower than the current index level, since the lookback is effectively a forward contract with zero volatility sensitivity. The fact that the theoretical delta of an at-the-money lookback call is negative – to hedge a long call position you must actually go long the index – is initially quite astonishing to market participants.

A similar effect holds for lookback puts, whose implied-tree deltas are also always numerically lower (that is, negative and larger in magnitude) than the corresponding Black-Scholes deltas.

APPENDIX: The relation between local and implied volatilities

In this appendix we provide some insight into our three rules of thumb. Our treatment is intuitive; for a more rigorous approach see Kani and Kamal [1996].

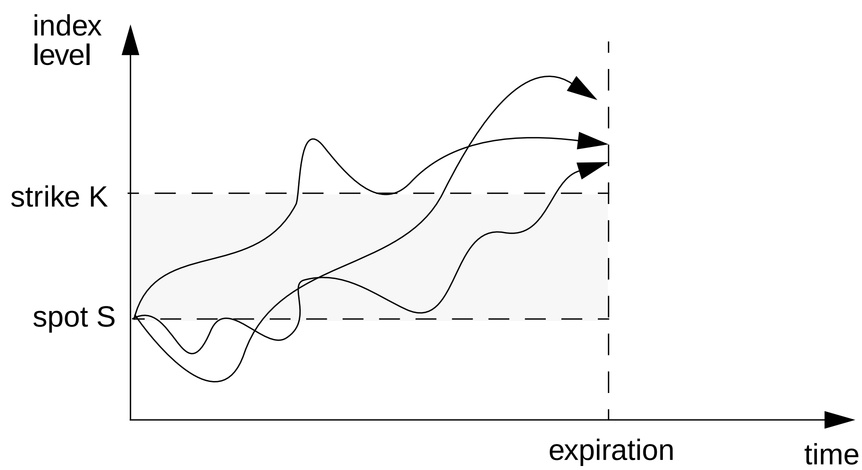
We restrict ourselves to the simple case in which the value of local volatility for an index is independent of future time, and varies linearly with index level, so that

$$\sigma(S) = \sigma_0 + \beta S \qquad \text{for all time t} \tag{A 1}$$

If you refer to the variation in future at-the-money local volatility as the "forward" volatility curve, then you can call this variation with index level the "sideways" volatility curve.

Consider the implied volatility $\Sigma(S,K)$ of a slightly out-of-the-money call option with strike K when the index is at S. Any paths that contribute to the option value must pass through the region between S and K, shown shaded in Figure 18. The volatility of these paths during most of their evolution is determined by the local volatility in the shaded region.

FIGURE 18. Index evolution paths that finish in the money for a call option with strike K when the index is at S. The shaded region is the volatility domain whose local volatilities contribute most to the value of the call option.



Because of this, you can roughly think of the implied volatility for the option of strike K when the index is at S as the average of the local volatilities over the shaded region, so that

$$\Sigma(S, K) \approx \frac{1}{K - S}\int_S^K \sigma(S')dS' \qquad \text{(A 2)}$$

By substituting Equation A1 into Equation A2 you can show that

$$\Sigma(S, K) \approx \sigma_0 + \frac{\beta}{2}(S + K) \qquad \text{(A 3)}$$

Equation A3 shows that, if implied volatility varies linearly with strike K at a fixed market level S, then it also varies linearly at the same rate with the index level S itself. This is Rule of Thumb 2 on page 16. Equation A1 then shows that local volatility varies with S at twice that rate, which is Rule of Thumb 1 on page 13. You can also combine Equation A1 and Equation A3 to write the relationship between implied and local volatility more directly as

$$\Sigma(S, K) \approx \sigma(S) + \frac{\beta}{2}(K - S) \qquad \text{(A 4)}$$

If $C_{BS}(S, \Sigma(S, K), r, t, K)$ represents the Black-Scholes formula for the value of a call option in the presence of an implied volatility surface $\Sigma(S, K)$, then its exposure is given by

$$\begin{aligned} \Delta = \frac{dC_{BS}}{dS} &= \frac{\partial C_{BS}}{\partial S} + \frac{\partial C_{BS}}{\partial \Sigma} \cdot \frac{\partial \Sigma}{\partial S} \\ &\approx \frac{\partial C_{BS}}{\partial S} + \frac{\partial C_{BS}}{\partial \Sigma} \cdot \frac{\partial \Sigma}{\partial K} \end{aligned} \qquad \text{(A 5)}$$

We have used the fact that $\frac{\partial \Sigma}{\partial S} \approx \frac{\partial \Sigma}{\partial K}$, a consequence of Equation A3, in writing the last identity. Equation A5 is equivalent to Rule of Thumb 3 on page 18.

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